Algebraic Notions of Non-Termination

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Abstract. We study and compare two notions of non-termination on idempotent semirings: infinite iteration and divergence. We determine them in various models and develop conditions for their coincidence. It turns out that divergence yields a simple and natural way of modelling infinite behaviour, whereas infinite iteration shows some anomalies.

1 Introduction

Idempotent semirings and Kleene algebras have recently been established as foundational structures in computer science. Initially conceived as algebras of regular expressions, they now find widespread applications ranging from program analysis and semantics to combinatorial optimisation and concurrency control.

Kleene algebras provide operations for modelling actions, programs or state transitions under non-deterministic choice, sequential composition and finite iteration. They have been extended by an omega operation for infinite iteration [7, 25], by domain and modal operators [10, 21] and by operators for program divergence [9]. The resulting formalisms bear strong similarities with propositional dynamic logics [13], but have a much richer model class that comprises relations, languages, paths, traces, automata and formal power series.

Among the most fundamental analysis tasks for programs are termination and non-termination. In a companion paper [9], different algebraic notions of termination based on modal semirings have been introduced and compared. The most important ones were the omega operator for infinite iteration [7] and the divergence operator which comprises the standard set-theoretic notion of well-foundedness. Although, intuitively, well-foundedness and absence of infinite iteration should be the same concept, it was found that they differ on some very natural models, including languages.

Here, we extend this investigation to the realm of non-termination. Our results confirm the anomalies of the omega operator beyond termination. They also suggest that the divergence semirings proposed in [9] are powerful tools that capture terminating and non-terminating behaviours on various standard models of programs and reactive systems. Our main contributions are as follows.

- We study infinite iteration and divergence in concrete models, including small finite examples, relations, traces, paths and languages. It turns out that these two concepts coincide in relation semirings; they are consistent

with the demonic view on total program correctness. However, they differ on all other models considered.
- We also study abstract *taming conditions* for omega that imply coincidence with divergence. We find a very heterogenous situation: Omega is tame on relation semirings. It is also tame on language semirings, but violates the taming condition. It is not tame on trace and path semirings. It shows all variants of behaviour already on three-element models.
- We introduce finite and infinite iteration in the context-free setting as fixed points of the function $\lambda x.b + ax$ and use some general results about fixed points, in particular fixed point fusion laws and the Knaster-Tarski theorem to relate them with Kleene algebras and omega algebras. We also determine conditions for iterating these fixed points.
- We use some standard techniques from universal algebra and Galois connections for constructing trace, path, language relation semirings and for relating these structures.

The remainder of the paper is organised as follows. Section 2 introduces idempotent semirings and notions of recursion on these structures. Section 3 shows how these general notions can be refined to Kleene algebras and omega algebras. Section 4 and Section 5 discuss circumstances when fixed points can be determined by iteration. Section 6 specialises this discussion to finite idempotent semirings and presents some examples. Section 7 introduces some properties of omega algebras for further calculations. Section 8 presents two examples that show the unexpected behaviour of omega. Sections 9 to 12 define trace, path, language and relation semirings and show some relationships between these models. Section 13 determines finite and infinite iteration in these models. Section 14 and 15 formalise a notion of divergence and determine divergences across models. Section 16 presents taming conditions for omega with respect to divergence. Section 17 sums up the results of this paper, discusses possibilities for future work and presents some open questions.

2 Idempotent Semirings with (Co-)Recursion

Our algebraic analysis of non-termination is based on idempotent semirings. Intuitively, semirings are rings without an additive inverse.

A *semiring* is a structure $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition from the left and right and 0 is a left and right zero of multiplication. A semiring S is *idempotent* (an *i-semiring*) if $(S, +)$ is a semilattice with $x + y = \sup(x, y)$. Concretely, we have the following axioms for i-semirings.

additive monoid: $a + (b + c) = (a + b) + c, \qquad a + 0 = a = 0 + a,$
commutativity: $a + b = b + a,$
idempotence: $a + a = a,$
multiplicative monoid: $a \cdot (b \cdot c) = (a \cdot b) \cdot c, \qquad a \cdot 1 = a = 1 \cdot a,$
distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c, \qquad (a + b) \cdot c = a \cdot c + b \cdot c,$

multiplicative zero: $a \cdot 0 = 0 = 0 \cdot a$.

Idempotent semirings are useful for modelling actions, programs or state transitions under non-deterministic choice and sequential composition. We usually omit the multiplication symbol. The semilattice-order $\leq$ on S has 0 as its least element; addition and multiplication are isotone with respect to it. Another useful concept is *semiring duality*, which holds between statements of a semiring and those of its *opposite* where the order of multiplication is swapped.

Tests of a program or sets of states of a transitions system can also be modelled in i-semirings. A *test* [18] in an i-semiring S is an element of a Boolean subalgebra $\mathsf{test}(S) \subseteq S$ (the *test algebra* of S) such that $\mathsf{test}(S)$ is bounded by 0 and 1 and multiplication coincides with lattice meet. We will write $a, b, c \dots$ for arbitrary semiring elements and $p, q, r, \dots$ for tests. Complementation will be denoted by $\neg$ and we will freely use the standard Boolean operations on tests with their usual laws.

Finite and infinite iteration can be modelled on an i-semiring S via fixed points of the "affine" mappings

$$\mathfrak{f}(x) = b + ax, \quad \mathfrak{g}(x) = ax \quad \text{and} \quad \mathfrak{a}(x) = 1 + ax$$

and their duals $\hat{\mathfrak{f}}(x) = b + xa$, $\hat{\mathfrak{g}}(x) = xa$, and $\hat{\mathfrak{a}}(x) = 1 + xa$ (with respect to opposition). As usual, the least pre-fixed point of a function f is given by $\inf(x : f(x) \leq x)$. It is denoted by μ_f. The greatest post-fixed points of a function f is given by $\sup(x : x \leq f(x))$ and denoted by ν_f. Since we are interested in some special cases we further abbreviate

$$\mu = \mu_{\mathfrak{f}}, \qquad \nu = \nu_{\mathfrak{f}}, \qquad a^* = \mu_{\mathfrak{a}} \quad \text{and} \quad a^\omega = \nu_{\mathfrak{g}}.$$

In these cases, the least pre-fixed points and the greatest post-fixed points are also least and greatest fixed points, respectively. Dual arguments apply to the fixed points of $\hat{\mathfrak{f}}$, $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{a}}$. They are denoted by $\hat{\mu}$, $\hat{\nu}$, $\hat{a}^*$ and $\hat{a}^\omega$.

Intuitively, a^* and $\hat{a}^*$ model tail recursions. Since multiplication is non-commutative, a^* and $\hat{a}^*$ can differ. The same argument holds for a^ω and $\hat{a}^\omega$, which model tail co-recursions. The fixed points μ, ν and their duals model more general forms of recursion or co-recursion.

We now analyse the fixed points of $\mathfrak{f}$ in more detail. A simple induction shows that an arbitrary fixed point ξ of $\mathfrak{f}$ satisfies, for all $n \in \mathbb{N}$, the recurrence

$$\xi = \mathfrak{f}^n(\xi) = a^n \xi + \sup(a^i b : 0 \leq i \leq n-1) \tag{1}$$

with $a^0 = 1$ and $a^{n+1} = aa^n$. Since the terms $\sup(a^i b : 0 \leq i \leq n-1)$ are essential for calculating fixed points we abbreviate $f^{(n)}(x) = \sup(f^i(x) : 0 \leq i \leq n)$ and $f^*(x) = \sup(f^i(x) : n \in \mathbb{N})$ for arbitrary functions f. The mapping $f^{(n)}(x)$ is isotone in n; it is also isotone in x whenever f is.

Identity (1) can now be written as $\xi = a^n \xi + \mathfrak{f}^{(n-1)}(0)$. A simple induction shows that

$$\mathfrak{f}^{(n)}(0) \leq \mu \qquad \text{and} \qquad \mathfrak{a}^{(n)}(0) \leq a^*$$

hold for all $n \in \mathbb{N}$ whenever μ and a^* exist, and therefore also

$$\mathfrak{f}^*(0) \le \mu \qquad \text{and} \qquad \mathfrak{a}^*(0) \le a^* .$$

As usual in fixed point theory, $\mathfrak{f}^*(0) = \mu$ and $\mathfrak{a}^*(0) = a^*$ need not hold; the elements μ and a^* need not even exist. We will later investigate conditions that enforce these equations, thus identifying recursion and iteration. To emphasise the bottom-up iteration by $f^{(n)}$ we sometimes write $f_\uparrow^{(n)}$.

Another interesting question is when μ reduces to tail recursion. Obviously, $\mathfrak{f}^{(n)}(0) = \mathfrak{a}^{(n)}(0)b$ holds for each $n \in \mathbb{N}$, but $\mathfrak{f}^*(0)$ need not be equal to $\mathfrak{a}^*(0)b$ unless the infinite distributivity law

$$\sup(a^n b : n \in \mathbb{N}) = \sup(a^n : n \in \mathbb{N})b$$

holds. Similarly, $\mathfrak{a}^{(n)}(0) = \hat{\mathfrak{a}}^{(n)}(0)$ and $\mathfrak{a}^*(0) = \hat{\mathfrak{a}}^*(0)$ hold but a^* and $\hat{a}^*$ can still be different.

Analogous formulas for ν and a^ω cannot even be written down due to the lack of a meet operation, which would be needed for top-down iteration, in semirings. We will present conditions for the existence of ν and a^ω in Section 4 and Section 5.

Concluding this discussion, we call $\mathfrak{f}$-*semiring* an i-semiring S in which the fixed points μ, $\hat{\mu}$, ν and $\hat{\nu}$ exist for all $a, b \in S$. In particular, setting $b = 0$ or $b = 1$ implies the existence of a^*, a^ω, $\hat{a}^*$ and $\hat{a}^\omega$.

3 Tail Recursions and Omega Algebras

We now investigate conditions for reducing μ to a tail recursion and for splitting ν into a recursive and a co-recursive part. We use fixed point fusion theorems (cf. [4]) to derive appropriate conditions. At the moment we only need their trivial parts.

Lemma 3.1. *Let f, g and h be functions on some poset.*

(*a*) $f \circ h \le h \circ g \Rightarrow \mu_f \le h(\mu_g)$, *whenever μ_f and μ_g exist.*
(*b*) $f \circ h \ge h \circ g \Rightarrow \nu_f \ge h(\nu_g)$, *whenever ν_f and ν_g exist.*

The non-trivial part of fixed point fusion deals with the converse inequalities. Its additional conditions will be explored later.

Lemma 3.2. *In every $\mathfrak{f}$-semiring,*

(*a*) 1^ω *is the greatest element,*
(*b*) $\mu \le a^* b$ *and, dually,* $\hat{\mu} \le b\hat{a}^*$.

Proof. (*a*) follows from the definition of a^ω; (*b*) from Lemma 3.1(*a*). ◻

Therefore, μ and $\hat{\mu}$ are tail recursive when the converse inequalities, $a^*b \leq \mu$ and its dual, are imposed. In order to enforce also $a^* = \hat{a}^*$, Leiß [19] has suggested the conditions

$$ba^* \leq \hat{\mu} \quad \text{and} \quad \hat{a}^*b \leq \mu.$$

These *Leiß conditions* imply that $a^*b = \mu$, $b\hat{a}^* = \hat{\mu}$ as well as $a^* = \hat{a}^*$ hold in $\mathfrak{f}$-semirings.

Lemma 3.3. *An $\mathfrak{f}$-semiring is a Kleene algebra if the Leiß conditions hold.*

Kleene algebras [17] are i-semirings that satisfy the *star unfold* and *star induction* axioms

$$1 + aa^* \leq a^*, \quad 1 + a^*a \leq a^*, \quad b + ac \leq c \Rightarrow a^*b \leq c, \quad b + ca \leq c \Rightarrow ba^* \leq c.$$

Lemma 3.4. *In every Kleene algebra, a^* formally models the reflexive transitive closure of a. It satisfies $1 + a + a^*a^* \leq a^*$ and $1 + a + bb \leq b \Rightarrow a^* \leq b$.*

It seems that Lemma 3.4 does not hold in all $\mathfrak{f}$-semirings. But we do not know a counterexample. We also do not know whether the reflexive transitive closure laws imply the star induction axioms in i-semirings. But we will later present sufficient conditions for this implication.

Let us now consider ν, where similar arguments apply.

Lemma 3.5. *In every $\mathfrak{f}$-semiring,*

(a) $\nu = \mu + a\nu$,
(b) $a^\omega + \mu \leq \nu$.

Proof.

(a) First we show that $\mu + a\nu$ is a fixed point of $\mathfrak{f}$. Since $b \leq \mu$ we obtain

$$\mathfrak{f}(\mu + a\nu) = b + a\mu + aa\nu = b + a\mu + ab + aa\nu = \mu + a(b + a\nu) = \mu + a\nu.$$

Now let ξ be another fixed point of $\mathfrak{f}$. Then $\xi = b + a\xi \leq \mu + a\nu$ follows from $\xi \leq \nu$ and $b \leq \mu$. Therefore $\mu + a\nu$ must be equal to ν.

(b) Set $f = \mathfrak{f}$, $g = \mathfrak{g}$ and $h = x + \mu$ in Lemma 3.1(b). □

As a consequence of Lemma 3.5(b), adding the condition

$$\nu \leq a^\omega + \mu \tag{2}$$

enforces $\nu = a^\omega + \mu$. This is similar to the Leiß conditions for the star.

Lemma 3.6. *A Kleene algebra is an omega algebra if condition (2) holds.*

An *omega algebra* [7] is a Kleene algebra that satisfies the *omega unfold* and the *omega co-induction* axiom

$$a^\omega \leq aa^\omega, \qquad c \leq b + ac \Rightarrow c \leq a^\omega + a^*b.$$

Here, the co-recursion of ν splits into a tail recursive and a tail co-recursive part, one of which models the finite and one that seems to models the infinite behaviour of ν.

Kleene algebras have been introduced for modelling finite iteration on an i-semiring; omega algebras are supposed to model infinite iteration as well. A particular strength of these approaches is that, in contrast to $\mathfrak{f}$-semirings, they allow first-order equational reasoning about these fixed points. Since i-semirings are equational classes, they are, by Birkhoff's HSP-theorem (cf. [26]), closed under subalgebras, direct products and homomorphic images. Since Kleene algebras and omega algebras are universal Horn classes, they are, by Mal'cev's quasi-variety theorem (cf. [26]), closed under subalgebras and direct products, but not in general under homomorphic images. These facts are useful for constructing new algebras from given ones. The possibility to finitely define Kleene algebras and omega algebras equationally is ruled out by the following two facts: First, Kleene algebras are complete for the equational theory of regular expressions [17]. Second, there is no finite equational axiomatisation for this theory [23].

The above discussion raises the question about the existence of μ and ν. We provide negative answers.

Lemma 3.7.

(a) *a^*, and therefore μ, does not exist in all i-semirings.*
(b) *a^ω, and therefore ν, does not exist in all Kleene algebras (even if there is a greatest element).*

Proof.

(a) It is well-known that the max-plus semiring is idempotent, but cannot be extended to a Kleene algebra (cf. [10]).
(b) We first present an example without and then a second example with a greatest element. Remember that this greatest element is, by Lemma 3.2, equal to 1^ω.
   (i) Consider $\mathbb{N}$ with addition and multiplication defined as max, except that $n0 = 0 = 0n$ for all $n \in \mathbb{N}$. This turns $\mathbb{N}$ into an i-semiring with 0 and 1 as neutral elements and which is ordered by $\mathbb{N}$. Since multiplication is idempotent, setting $n^* = \max(1, n)$ turns $\mathbb{N}$ into a Kleene algebra. In general, we can approximate multiplication by $nm \leq \max(n, m)$ for all $m, n \in \mathbb{N}$.
   Verifying the star unfold axiom $1 + nn^* \leq n^*$ is straightforward by $1 + nn^* \leq \max(1, n, n^*) = \max(1, n, 1, n) = \max(1, n) = n^*$.
   We now verify the star induction axiom $l + mn \leq n \Rightarrow m^*l \leq n$, or equivalently, that $\max(l, mn) \leq n$ implies $\max(l, 1, m) \leq n$. If $n \geq 1$, the assumption is $\max(l, m, n) \leq n$, which implies the claim since $\max(l, 1, m) \leq \max(l, m, n)$. In the case of $n = 0$ the assumption implies $l = 0$ and the claim reduces to $0 \leq 0$.
   The dual star unfold and star induction axioms follow immediately from commutativity of multiplication. However, $\mathbb{N}$ does not possess a greatest element, so by Lemma 3.2(a) 1^ω is undefined.

(ii) Consider now the set of all finite non-empty subsets of $\mathbb{N} \cup \{\infty\}$ with addition defined as elementwise max, i.e., $M + N = \{\max(m, n) : m \in M, n \in N\}$ and multiplication as elementwise min. This forms an i-semiring with $\{0\}$ and $\{\infty\}$ as neutral elements and $\{\infty\}$ as greatest element. It becomes a Kleene algebra by setting $N^* = \{\infty\}$ for any finite, non-empty subset N. We show by contradiction that N^ω does not exist for $N = \{0, \infty\}$. Let ξ be the greatest fixed point of $N \cdot X$. Since obviously $\xi \neq \{\infty\}$ there is a number $n_0 \in \mathbb{N}$ with $n_0 > n$ for all $n \in \xi - \{\infty\}$. Now $\zeta = \xi \cup \{0, n_0\}$ is also a fixed point of $N \cdot X$, since $N \cdot \zeta = (N \cdot \xi) \cup (N \cdot \{0, n_0\}) = \xi \cup \{0, n_0\} = \zeta$. Furthermore, $\xi + \zeta = \zeta$ and therefore $\xi \leq \zeta$. Since $\xi \neq \zeta$ this contradicts the assumption. □

4 (Co-)Continuity and Iteration

We have seen that μ and ν can, under certain circumstances, be reduced to tail recursion and tail co-recursion. Moreover, $\mathfrak{f}_\uparrow^{(n)}(0) \leq \mathfrak{f}_\uparrow^*(0) \leq \mu$. Obviously, we can also define the functions

$$f_\downarrow^{(n)}(x) = \inf(f^n(x) : 0 \leq i \leq n) \qquad \text{and} \qquad f_\downarrow^*(x) = \inf(f^n(x) : n \in \mathbb{N}).$$

Then $\nu \leq \mathfrak{f}_\downarrow^*(\top) \leq \mathfrak{f}_\downarrow^{(n)}(\top)$ whenever $\top$ and the necessary infima exist.

We now investigate conditions under which μ becomes an iteration and ν a co-iteration, which implies that these fixed points exist. This uses the non-trivial variants of fixed point fusion (cf. [4]). We provide a proof since we need only a particular instance.

Theorem 4.1. *Let f, g and h be functions on some bounded poset. Let also $\mu_g = g_\uparrow^*(0)$, $\mu_f = f_\uparrow^*(0)$, $\nu_g = g_\downarrow^*(\top)$ and $\nu_f = f_\downarrow^*(\top)$.*

(*a*) $f \circ h \geq h \circ g \Rightarrow \mu_f \geq h(\mu_g)$ *if h distributes over arbitrary suprema of $g^n(0)$.*
(*b*) $f \circ h \leq h \circ g \Rightarrow \nu_f \leq h(\nu_g)$ *if h distributes over arbitrary infima of $g^n(0)$.*

Proof. We only show the proof of (a), that of (b) is dual. Note that the assumption implies that $h(0) = 0$.

$$\begin{aligned} h(\mu_g) &= h(\sup(g^n(0) : n \in \mathbb{N})) \\ &= \sup(h(g^n(0)) : n \in \mathbb{N}) \\ &\leq \sup(f^n(h(0)) : n \in \mathbb{N}) \\ &= \sup(f^n(0) : n \in \mathbb{N}) \\ &= \mu_f . \end{aligned}$$

□

Fixed point fusion immediately links μ with tail iteration.

Corollary 4.2. *Let* $\sup(\mathfrak{a}^n(0) : n \in \mathbb{N})b = \sup(\mathfrak{a}^n(0)b : n \in \mathbb{N})$. *Then* $\hat{a}^*b \leq \mu$ *holds.*

A dual statement links the dual infinite distributivity law with the dual condition $ba^* \leq \hat{\mu}$. Therefore $\mu = a^*b$ and $\hat{\mu} = \hat{a}^*$.

It remains to link a^* with $\hat{a}^*$. This additional condition can be enforced by combining the previous conditions into the $*$-*continuity* axiom

$$ab^*c = \sup(ab^nc : n \in \mathbb{N}).$$

Lemma 4.3. *In every $*$-continuous $\mathfrak{f}$-semiring $\mu = a^*b$, $\hat{\mu} = ba^*$ and $a^* = \hat{a}^*$ hold.*

Proof. This is immediate from fixed point fusion (Lemma 3.1 and Corollary 4.2). □

Corollary 4.4. *Every $*$-continuous i-semiring is a Kleene algebra.*

Star continuous Kleene algebras have been introduced and studied in [16]. μ and $\hat{\mu}$ can now be determined iteratively as

$$\mu = a^*b = \mathfrak{a}^*(0)\ b \quad \text{and} \quad \hat{\mu} = ba^* = b\ \mathfrak{a}^*(0).$$

For a similar treatment of omega, we call ω-*co-continuity* axiom the expression

$$a^\omega + a^*b = \inf(a^n\top + a^*b : n \in \mathbb{N}). \tag{3}$$

This definition implies that all necessary infima exist.

Lemma 4.5. *$\nu = a^\omega + a^*b$ holds in every $*$-continuous and ω-co-continuous i-semiring.*

Proof. By fixed point fusion. □

Corollary 4.6. *Every $*$-continuous and ω-co-continuous i-semiring is an omega algebra.*

As already mentioned, $*$-continuity and ω-co-continuity assume the existence of certain suprema and infima and certain infinite distributivity laws. Without this implicit distributive law, $a^* = \mathfrak{a}^*(0)$ alone would not suffice to subsume the Kleene algebra axioms.

5 Completeness and Iteration

A second way to guarantee the existence of μ and ν is to assume a complete semilattice reduct in the i-semiring. We will briefly call such structures *complete i-semirings*. It is well-known that every complete semilattice S is also a complete lattice with, for all $X \subseteq S$,

$$\inf(X : X \subseteq S) = \sup(y \in S : \exists x \in X.y \leq x).$$

Lattices with an additional multiplicative monoid structure and the usual distributive laws for multiplication and addition are known as *lattice-ordered monoids* (cf. [6]).

Lemma 5.1. *Every complete i-semiring is also a complete lattice-ordered monoid.*

Existence of μ and ν on complete lattice-ordered monoids follows, as usual, from the Knaster-Tarski theorem. Both fixed points can be determined iteratively as

$$\mu = \mathfrak{f}_{\uparrow}^{*}(0) \quad \text{and} \quad \nu = \mathfrak{f}_{\downarrow}^{*}(\top).$$

This situation is similar to that in the last section.

Lemma 5.2. *Every complete i-semiring can be extended to an $\mathfrak{f}$-semiring.*

Moreover, $a^* = \mathfrak{a}_{\uparrow}^{*}(0) = \hat{a}^*$ and $a^\omega = \mathfrak{g}_{\downarrow}^{*}(\top) = \inf(a^i\top : n \in \mathbb{N})$.

But what about the reduction of μ to tail recursion and and the splitting of ν? We consider again the Leiß conditions and (2). Obviously, applying fixed point fusion for μ requires that multiplication distributes over arbitrary suprema. However, in complete i-semirings, the infinite distributivity laws

$$a \sup(b_i) = \sup(ab_i) \quad \text{and} \quad \sup(a_i)b = \sup(a_i b) \tag{4}$$

need not hold for an arbitrary set I and $i \in I$.

Lemma 5.3. *The Leiß conditions hold in every complete i-semiring with infinite distributivity laws (4). Every such semiring is $*$-continuous.*

Similarly, fixed point fusion for ν requires that ν (additively) distributes over arbitrary infima. Again, in complete i-semirings, these infinite distributivity laws

$$a + \inf(b_i) = \inf(a + b_i) \tag{5}$$

need not hold for $i \in I$.

Lemma 5.4. *Identity (2) holds in every complete i-semiring with infinite distributivity law (5). Every such semiring is ω-co-continuous.*

Lemma 5.5. *Every complete $\mathfrak{f}$-semiring that satisfies the infinite distributivity laws (4) and (5) is an omega algebra.*

By the discussion of the previous section, under these circumstances, $\mathfrak{f}$-semirings reduce to Kleene algebras and omega algebras. Moreover, the iterative definition of a^* as the reflexive transitive closure of a now suffices to link a^* and μ by tail recursion and to subsume the Kleene algebra axioms.

Lemma 5.6. *In the class of complete i-semirings with infinite distributivity law (5), the star unfold and star induction axioms hold and are equivalent to the reflexive-transitive closure axioms from Lemma 3.4.*

The infinite distributivity laws hold a priori when the lattice reduct of the i-semiring is a complete Boolean algebra. In this case, $\lambda x.ax$ and $\lambda x.xa$ are lower adjoints of the Galois connections defining residuals. They therefore distribute with arbitrary suprema. Similarly, $\lambda x.a + x$ is an upper adjoint of the Galois connection defining Boolean difference. It therefore distributes with arbitrary infima.

6 Finite Idempotent Semirings

Finite i-semiring enjoy all the properties of the previous sections. First, they are a fortiori complete and therefore all fixed points under consideration exist.

Lemma 6.1. *Every finite i-semiring is also a complete lattice-ordered monoid.*

Second, they satisfy all necessary distributivity laws. Fixed point fusion therefore always applies; recursion and co-recursion reduce to their tail variants.

Lemma 6.2. *Every finite i-semiring can uniquely be extended to a $*$-continuous Kleene algebra and to a ω-co-continuous omega algebra.*

Third, by putting things together, all fixed points can be determined iteratively. So again $a^* = \mathfrak{a}_\uparrow^*(0) = \hat{a}^*$, $a^\omega = \mathfrak{g}_\downarrow^*(\top)$ and the remaining fixed points can be determined from these as $\mu = a^*b$, $\hat{\mu} = ba^*$ and $\nu = a^\omega + \mu$.

Corollary 6.3. *Every finite $\mathfrak{f}$-semiring is an omega algebra.*

As a consequence of Lemma 5.6, the reflexive-transitive closure axioms are now strong enough to enforce Kleene algebras.

Lemma 6.4. *The star unfold and star induction axioms hold in the class of finite i-semirings. They are equivalent to the reflexive-transitive closure axioms from Lemma 3.4.*

Due to the finite size of the lattice all iterations become stationary after finitely many steps. Since $f_\uparrow^{(n)}$ is isotone and $f_\downarrow^{(n)}$ is antitone in n, the iterations that determine $f_\uparrow^*$ and $f_\downarrow^*$ follow chains in the lattice. Upper bounds for iteration are therefore given by κ, the length of the longest chain in the lattice, that is, $f_\uparrow^*(x) = f_\uparrow^{(\kappa)}(x)$ and $f_\downarrow^*(x) = f_\downarrow^{(\kappa)}(x)$, since iterations then become stationary. In particular $a^* = \mathfrak{a}_\uparrow^{(\kappa)}(0)$ and $a^\omega = \mathfrak{g}_\downarrow^{(\kappa)}(\top)$. The size of the i-semiring S yields a less tight bound.

While the computations of a^* and μ are immediate from the addition and multiplication tables, those of a^ω and ν require the computations of meets in the semiring. They can be computed from the *transitive reduct* of $\leq$, which is a least relation r with transitive closure $\leq$. Then $\inf(a,b) = r^\circ(a) \cap r^\circ(b)$, where r° denotes the converse of r and $r(a)$ the relational image of a under r.

We have explicitly computed the stars and omegas for some small finite models with William McCune's Mace4 tool [1]. The specification of i-semirings and omega algebras for Mace4 and the associated automated deduction system Prover9 can be found in Appendix B. These examples have already been listed in [8], but with a different axiomatisation for the star. The omega has not been considered so far.

Example 6.5. The two-element Boolean algebra is also an i-semiring and an omega algebra with $0^* = 1^* = 1^\omega = 1$ and $0^\omega = 0$. It is the only two-element omega algebra.

Example 6.6. There are three three-element i-semirings. Their elements are from $\{0, a, 1\}$. Mace4 shows that $0^* = 1^* = 1$, $0^\omega = 0$ and $1^\omega = \top$ holds in all models; only a is free in the defining tables.

(a) In A_3^1, addition is defined by $0 < 1 < a$. The remaining operations are given by $aa = a^* = a^\omega = a$.
(b) In A_3^2, addition is defined by $0 < a < 1$. The remaining operations are given by $aa = a^\omega = 0$ and $a^* = 1$.
(c) In A_3^3, addition is defined by $0 < a < 1$. The remaining operations are given by $aa = a^\omega = a$ and $a^* = 1$.

Beyond these results, Mace4 generated 20 i-semirings and omega algebras with four, 149 with five, 1488 with six and 18554 with seven elements. Tables for all omega algebras with up to four elements are listed in Appendix C. In contrast, Conway's book lists 21 i-semirings with four elements. But his examples (5.) and (7.) are wrong, and one is missing. The numbers for i-semirings and omega algebras with more than five elements should be taken with a grain of salt. McCune writes that the isomorphism checker that comes with Mace4 "eliminates the isomorphic [models, but] does not attempt to permute operations when checking for isomorphism. For example, [...] dual lattices are not necessarily isomorphic." [1]. However, the numbers up to dimension 6 are confirmed by Jipsen's computations [14].

7 Properties of Star and Omega

Kleene algebras are sound and complete for the equational theory of regular expressions [17]. Therefore, all regular identities hold in Kleene algebra and we will freely use them. Examples are $0^* = 1 = 1^*$, $1 \leq a^*$, $aa^* \leq a^*$, $a^*a^* = a^*$, $a \leq a^*$, $a^*a = aa^*$ and $1 + aa^* = a^* = 1 + a^*a$. Furthermore the star is isotone.

It has also been shown that ω-regular identities such as $0^\omega = 0$, $a \leq 1^\omega$, $a^\omega = a^\omega 1^\omega$, $a^\omega = aa^\omega$, $a^\omega b \leq a^\omega$, $a^*a^\omega = a^\omega$ and $(a + b)^\omega = (a^*b)^\omega + (a^*b)^*a^\omega$ hold in omega algebras [7] and that omega is isotone. Furthermore, since every omega algebra is a $\mathfrak{f}$-semiring, it has a greatest element, namely $\top = 1^\omega$ (cf. Lemma 3.2(a)). However, omega algebras are not complete for the equational theory of ω-regular expressions: Products of the form ab exist in ω-regular languages only if a represents a set of finite words whereas no such restriction is imposed on omega algebra terms.

The following Lemma will be used for analysing omegas in concrete models by splitting an action into separate parts.

Lemma 7.1. *Let a, b, c be elements of some i-semiring and let p be a test.*

(a) $(a + p)a^*p = a^*p$.
(b) $(a + p)^*p = a^*p$.
(c) $a^\omega = b^\omega + (b^*c)a^\omega$ *holds for* $a = b + c$.
(d) $(a + p)^\omega = a^\omega + a^*p\top$.
(e) $p^\omega = p\top$.

Proof.

(a) By multiplicative idempotence of p and some regular identities,

$$(a+p)a^*p = aa^*p + pa^*p = aa^*p + pp + paa^*p = (1+aa^*)p = a^*p.$$

(b) Direction ($\geq$) holds by isotonicity, whereas ($\leq$) is immediate from (a) by star induction.

(c) Let $\alpha = (b^*c)$. Then $a^\omega = (b+c)^\omega = \alpha^\omega + \alpha^* b^\omega$ is an ω-regular identity. Consequently,

$$a^\omega = \alpha^\omega + \alpha^* b^\omega = \alpha\alpha^\omega + (1+\alpha\alpha^*)b^\omega = b^\omega + \alpha(\alpha^\omega + \alpha^* b^\omega) = b^\omega + \alpha a^\omega.$$

(d) $(a+p)^\omega \leq a^\omega + a^*p\top$ follows from (a) and $(a+p)^\omega \leq \top$.
For the converse inequality it suffices to show that $a^\omega + a^*p\top$ is a fixed point of $\lambda x.(a+p)x$. By (a) and omega unfold,

$$(a+p)(a^\omega + a^*p\top) = aa^\omega + pa^\omega + aa^*p\top + pa^*p\top = a^\omega + a^*p\top.$$

(e) Set $a = 0$ in (d). $\square$

8 Star and Omega in Two Examples

We will first illustrate the computations of star and omega in a simple finite relational example.

Example 8.1. Consider the binary relation $a = \{(p,q),(q,r),(r,q),(p,s)\}$ over $P = \{p,q,r,s\}$. It is depicted in the left-most graph in Figure 1. The greatest

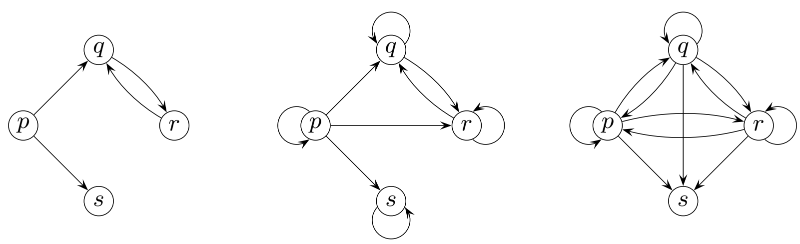

Fig. 1. The relations a, a^* and a^ω.

element in set of all binary relations over P is $\top = P \times P$ and the least element is $\emptyset$. We have used the RelView System [2] for computing star and omega. This system has been developed and implemented by Rudolf Berghammer and Ulf Milanese. The code on which these computations are based can be found in Appendix A.

By Lemma 3.4, a^* is the reflexive transitive closure of a. That is, a^* represents the finite a-paths by collecting their input and output points: $(x,y) \in a^*$ iff there is a finite a-path from x to y. Iterating $a^* = \mathfrak{a}^*(0) = \sup(a^i : i \in \mathbb{N})$ with RelView yields

$$a^* = \{(p,p),(p,q),(p,r),(p,s),(q,q),(q,r),(r,r),(r,q),(s,s)\}.$$

The relation a^* is represented by the second graph in Figure 1.

But what about a^ω? One might expect that it represents infinite a-paths in the sense that $(x,y) \in a^\omega$ iff there is an infinite a-path between x and y or iff x and y lie on an infinite a-path. However, iterating $a^\omega = \mathfrak{g}^*_\downarrow(\top) = \inf(a^i\top : i \in \mathbb{N})$ with RelView yields

$$a^\omega = \{(p,p),(p,q),(p,r),(p,s),(q,p),(q,q),(q,r),(q,s),(r,p),(r,q),(r,r),(r,s)\}.$$

The relation a^ω is represented by the right-most graph in Figure 1. Obviously, $(q,p) \in a^\omega$ although there is no a-path from q to p, neither finite nor infinite.

So what does a^ω represent? Let ∇a model those nodes from which a diverges, i.e., from which an infinite a-path emanates. Then Example 8.1 shows that elements in ∇a are linked by a^ω to any other node; elements outside of ∇a are not in the domain of a^ω. Interpreting a^ω generally as *anything for states on which a diverges* would be consistent with the demonic semantics of total program correctness; its interpretation of *nothing for states on which a diverges* models partial correctness. This further suggests to investigate the properties

$$(\nabla a)\top = a^\omega \quad \text{and} \quad \nabla a = \mathsf{dom}(a^\omega).$$

These two identities do not only hold in Example 8.1; they will be of central interest in this paper. To study them further, we will now introduce some important models of i-semirings and then formalise divergence in this setting.

We now revisit an example that has already been used to show that omega differs from the standard set-theoretic notion of well-foundedness [9].

Example 8.2. Consider the set

$$S = \{(n, n+k) : n, k \in \mathbb{N}\}$$

of all pairs of natural numbers where the first element is not greater than the second one. It can be shown that 2^S forms a complete i-semiring under the usual relational operations. In particular, S is its greatest and $\emptyset$ its least element. Therefore, by Lemma 5.2, star and omega exist and can be determined by iteration.

Consider now the relation $a = \{(n, n+1) : n \in \mathbb{N}\} \in 2^S$. Then

$$a^\omega = \inf(a^i S : i \in \mathbb{N}) = \inf(\{(n, n+k) : k \geq i\} : i \in \mathbb{N}) = \emptyset,$$

since no pair $(n, n+j)$, for arbitrary j, will survive iteration $j+1$ and therefore no pair will be present in the infimum.

However, a^ω is not well-founded (or, more precisely, *Noetherian*) in the standard set-theoretic sense. In the context of the previous discussion, ∇a should therefore not vanish.

As a conclusion, Example 8.1 suggests that a^ω models *anything* for states on which it diverges, whereas Example 8.2 shows that it models *nothing* for states on which it diverges. So what does a^ω model then?

We propose a partial answer to this question after studying further models.

9 Trace Semirings

In the next four sections we introduce some of the most interesting models of i-semirings: traces, paths, languages and relations. We will study divergence and omega on these models afterwards.

As usual, a *word* over a set Σ is a mapping $[0..n] \to \Sigma$. The empty word is denoted by ε and *concatenation* of words σ_0 and σ_1 by $\sigma_0.\sigma_1$. We write $\mathsf{first}(\sigma)$ for the first element of a word σ and $\mathsf{last}(\sigma)$ for its last element. We write $|\sigma|$ for the length of σ. The set of all words over Σ is denoted by Σ^*.

A (finite) *trace* over the sets P and A is either ε or a word σ such that $\mathsf{first}(\sigma), \mathsf{last}(\sigma) \in P$ and in which elements from P and A alternate. We will use $\tau_0, \tau_1, \ldots$ for denoting traces. The product of traces τ_0 and τ_1 is the trace

$$\tau_0 \cdot \tau_1 = \begin{cases} \sigma_0.p.\sigma_1 & \text{if } \tau_0 = \sigma_0.p \text{ and } \tau_1 = p.\sigma_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Intuitively, $\tau_0 \cdot \tau_1$ glues two traces together when the last state of τ_0 and the first state of τ_1 are equal. It then follows that $\mathsf{first}(\tau_0 \cdot \tau_1) = \mathsf{first}(\tau_0)$ and $\mathsf{last}(\tau_0 \cdot \tau_1) = \mathsf{last}(\tau_1)$ whenever this product exists. The set of all traces over P and A is denoted by $(P, A)^*$. Traces naturally arise in the context of labelled transition systems [5] and as an abstract interpretation for program schemes [15].

Lemma 9.1. *The power-set algebra $2^{(P,A)^*}$ with addition defined by set union, multiplication by $T_0 \cdot T_1 = \{\tau_0 \cdot \tau_1 : \tau_0 \in T_0, \tau_1 \in T_1$ and $\tau_0 \cdot \tau_1$ defined$\}$, P as unit and $\emptyset$ as zero is an i-semiring.*

We call this i-semiring the *full trace semiring* over P and A. By definition, $T_0 \cdot T_1 = \emptyset$ if all products between traces in T_0 and traces in T_1 are undefined. Full trace semirings admit rich test algebras: 2^P, for instance, is a Boolean algebra by definition.

Every subalgebra of the full trace semiring is, by the HSP-theorem, again an i-semiring. All constants such as 0, 1 and $\top$ are fixed by the subalgebra construction. We will henceforth consider only complete subalgebras of full trace semirings and call them *trace semirings*. Every non-complete subalgebra of the full trace semiring can of course uniquely be closed to a complete subalgebra.

10 Path Semirings

As we will see, forgetting parts of the structure is quite useful. First we want to forget all actions of traces. Consider the *projection* $\phi_P : (P, A)^* \to P^*$ which is defined, for all $p \in P$ and $a \in A$ by

$$\phi_P(\varepsilon) = \varepsilon, \qquad \phi_P(p.\sigma) = p.\phi_P(\sigma), \qquad \phi_P(a.\sigma) = \phi_P(\sigma).$$

ϕ_P is a mapping between traces and words over P which we call *paths*. A product on paths can be defined as for traces. For paths π_0 and π_1,

$$\pi_0 \cdot \pi_1 = \begin{cases} \sigma_0.p.\sigma_1 & \text{if } \pi_0 = \sigma_0.p \text{ and } \pi_1 = p.\sigma_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Again, $\pi_0 \cdot \pi_1$ glues two paths together when the last state of π_0 and the first state of π_1 are equal.

The mapping ϕ_P can be extended to a set-valued mapping $\phi_P : 2^{(P,A)^*} \to 2^{P^*}$ by taking the image, i.e.,

$$\phi_P(T) = \{\phi_P(\tau) : \tau \in T\}.$$

Now, ϕ_P sends sets of traces to sets of paths.

The information about actions can be introduced to paths by *fibration*, which can be defined in terms of the relational inverse $\phi_P^{-1} : P^* \to 2^{(P,A)^*}$ of ϕ_P. Intuitively, it fills the spaces between states in a path with all possible actions and therefore maps a single path to a set of traces. The mapping ϕ_P^{-1} can as well be lifted to the set-valued mapping

$$\phi_P^\sharp(Q) = \sup(\phi_P^{-1}(\pi) : \pi \in Q),$$

where $Q \in 2^{P^*}$ is a set of paths.

Lemma 10.1. *ϕ_P and $\phi_P^\sharp$ are adjoints of a Galois connection,*

$$\phi_P(a) \leq b \Leftrightarrow a \leq \phi_P^\sharp(b).$$

The proof is straightforward. Galois connections are interesting because they give theorems for free (cf. [3]). In particular, ϕ_P commutes with all existing suprema and $\phi_P^\sharp$ commutes with all existing infima. ϕ_P is isotone and $\phi_P^\sharp$ is antitone. Both mappings are related by the cancellation laws $\phi_P \circ \phi_P^\sharp \leq id_{2^{P^*}}$ and $id_{2^{(P,A)^*}} \leq \phi_P^\sharp \circ \phi_P$. Finally, the mappings are pseudo-inverses, i.e, $\phi_P \circ \phi_P^\sharp \circ \phi_P = \phi_P$ and $\phi_P^\sharp \circ \phi_P \circ \phi_P^\sharp = \phi_P^\sharp$.

Lemma 10.2. *The mappings ϕ_P are homomorphisms.*

Proof. We first consider $\phi_P : (P, A)^* \to P^*$. Then $\phi_P(\tau_0 \cdot \tau_1) = \phi_P(\tau_0) \cdot \phi_P(\tau_1)$ and $\phi_P(\varepsilon) = \varepsilon$ are immediate from the definition of trace and path products.

Therefore $\phi_P(T_0 \cdot T_1) = \phi_P(T_0) \cdot \phi_P(T_1)$ as well for sets of traces T_0 and T_1. Moreover, $\phi_P(T_0 + T_1) = \phi_P(T_0) + \phi_P(T_1)$ and $\phi_P(\emptyset) = \emptyset$) follow from the Galois connection and $\phi_P(P) = P$ holds by definition. ◻

By the HSP-theorem the set-valued homomorphism induces path semirings from trace semirings.

Lemma 10.3. *The power-set algebra* 2^{P^*} *is an i-semiring.*

We call this i-semiring *full path semiring* over P. It is the homomorphic image of a full trace semiring. Again, by the HSP-theorem, all subalgebras of full paths semirings are i-semirings; complete subalgebras are called *path semirings.*

Lemma 10.4. *Every identity that holds in all trace semirings holds in all path semirings.*

Moreover, the class of trace semirings contains isomorphic copies of all path semirings. This can be seen as follows.

Consider the congruence $\sim_P$ on a trace semiring over P and A that is induced by the homomorphism ϕ_P. The associated equivalence class $[T]_P$ contains all those sets of traces that differ in actions, but not in paths. From each equivalence class we can chose a special canonical representative, which is a set of traces that are built from one single action. Each of this representative is of course equivalent to a set of paths and therefore an element of a path semiring. Conversely, every element of a path semiring can be expanded to an element of some trace semiring by filling in the same action between all states.

The following lemma can be proved using standard techniques from universal algebra.

Lemma 10.5. *Let* S *be the full trace semiring over* P *and* A. *The quotient algebra* $S/{\sim_P}$ *is isomorphic to each full trace semiring over* P *and* $\{a\}$ *with* $a \in A$ *and to the full path semiring over* P*:*

$$S/{\sim_P} \cong 2^{(P,\{a\})^*} \cong 2^{P^*}.$$

In particular, the mappings ϕ_P and $\phi_P^\sharp$ are isomorphisms between the full trace semiring $2^{(P,\{a\})^*}$ and the full path semiring 2^{P^*}. In that case, $\phi_P^\sharp$ is not only the pseudo-inverse of ϕ_P, it is the inverse function of ϕ_P and vice versa, that is $\phi_P^{-1} = \phi_P^\sharp$.

Lemma 10.5 is not only restricted to full trace and path semirings. It immediately extends to trace and path semirings, since the operations of forming subalgebras and of taking homomorphic images always commute. In particular, each path semiring is isomorphic to some trace semiring with a single action. This isomorphic embedding of path semirings into the class of trace semirings implies the following proposition.

Proposition 10.6. *Every property that holds in all trace semirings holds in all path semirings.*

11 Language Semirings

Instead of forgetting actions we will now forget all states of traces to obtain language semirings. Most of the arguments of the last section will still hold, but not all.

Consider the projection $\phi_L : (P, A)^* \to A^*$ which is defined, for all $p \in P$ and $a \in A$, by

$$\phi_L(\varepsilon) = \varepsilon, \qquad \phi_L(p.\sigma) = \phi_L(\sigma), \qquad \phi_L(a.\sigma) = a.\phi_L(\sigma).$$

Now ϕ_L is maps traces to *words* over A. As mentioned in Section 9, the product on words is defined in the standard way as concatenation. Therefore, the product on words is a total function, whereas the one on traces is partial.

Again, the mapping ϕ_L can be extended to a set-valued mapping $\phi_L : 2^{(P,A)^*} \to 2^{A^*}$ by taking the image, i.e., $\phi_L(T) = \{\phi_L(\tau) : \tau \in T\}$. Now, ϕ_L sends sets of traces to *languages*. Similar to $\phi_P^\sharp$, information about states can be introduced to words by fibration, which is defined in terms of the relational inverse $\phi_L^{-1} : A^* \to 2^{(P,A)^*}$ of ϕ_L. Intuitively, it fills the spaces before and after actions in a word with all possible states and therefore maps a single word to a set of traces. The mapping ϕ_L^{-1} also be lifted to a set-valued mapping $\phi_L^\sharp(L) = \sup(\phi_L^{-1}(w) : w \in L)$, for any language $L \in 2^{L^*}$. Again there is a relationship between the two mappings.

Lemma 11.1. *ϕ_L and $\phi_L^\sharp$ are adjoints of a Galois connection.*

Therefore, ϕ_L and $\phi_L^\sharp$ share the properties of ϕ_P and $\phi_P^\sharp$ from the previous section. However, both mappings ϕ_P do not preserve multiplication.

Lemma 11.2. *The mappings ϕ_P are not homomorphisms.*

Proof. The product $\tau_0 \cdot \tau_1$ is undefined for $\tau_0 = pap$ and $\tau_1 = qap$. Therefore $\phi_L(\tau_0 \cdot \tau_1)$ is undefined as well, but $\phi_L(\tau_0).\phi_L(\tau_1) = a.a$. This extends to the set-valued case by taking $T_0 = \{pap\}$ and $T_1 = \{qap\}$. □

Note that $\phi_L(T_0 \cdot T_1) \subseteq \phi_L(T_0).\phi_L(T_1)$, but not in general conversely.

Here, we cannot use the HSP-theorem together with the set-valued homomorphism to introduce language semirings from trace semirings. Nevertheless the following fact is well-known.

Lemma 11.3. *The power-set algebra 2^{A^*} is an i-semiring.*

We call this i-semiring *full language semiring* over A. Again, by the HSP-theorem, all subalgebras of full language semirings are i-semirings; complete subalgebras are called *language semirings*.

Still, the class of trace semirings contains isomorphic copies of all language semirings.

Lemma 11.4. *Let S be the full trace semiring over $\{p\}$ and A. Then S is isomorphic to the full path semiring over A:*

$$2^{(\{p\},A)^*} \;\cong\; 2^{A^*}.$$

We could still define an equivalence relation $\equiv_L$ by partitioning the class of trace semirings according to sets of traces that differ only on states. However, it can be shown along the lines of the proof of Lemma 11.2 that this equivalence is not a congruence and therefore the quotient structure is not a semiring.

At least, the mappings ϕ_P and $\phi^\sharp_P$ are isomorphisms between the full trace semiring $2^{(\{p\},A)^*}$ and the full path semiring 2^{A^*}.

Lemma 11.4 can again be extended to (non-full) trace and language semirings; each language semiring is isomorphic to some trace semiring with one single state. This isomorphic embedding of language semirings into the class of trace semirings implies the following proposition.

Proposition 11.5. *Every property that holds in all trace semirings holds in all language semirings.*

12 Relation Semirings

Now we forget entire paths between the first and the last state of a trace. We therefore consider the mapping $\phi_R : (P, A)^* \to P \times P$ defined by

$$\phi_R(\tau) = \begin{cases} (\mathsf{first}(\tau), \mathsf{last}(\tau)) & \text{if } \tau \neq \varepsilon, \\ \text{undefined} & \text{if } \tau = \varepsilon. \end{cases}$$

It sends trace products to (standard) relational products on pairs. As before, ϕ_R can be extended to a set-valued mapping $\phi_R : 2^{(P,A)^*} \to 2^{P \times P}$ by taking the image, i.e., $\phi_R(T) = \{\phi_R(\tau) : \tau \in T\}$. Now, ϕ_R sends sets of traces to *relations*. Information about the traces between starting and ending state can be introduced to pairs of states by the fibration $\phi_R^{-1} : P \times P \to 2^{(P,A)^*}$ of ϕ_P. Intuitively, it replaces a pair of states by all possible traces between them. It can again be lifted to the set-valued mapping $\phi^\sharp_R(R) = \sup(\phi_R^{-1}(r) : r \in R)$, for any relation $R \in 2^{P \times P}$.

Lemma 12.1. *ϕ_R and $\phi^\sharp_R$ are adjoints of a Galois connection.*

The standard properties hold again.

Lemma 12.2. *The mappings ϕ_P are homomorphisms.*

By the HSP-theorem the set-valued homomorphism induces relation semirings from trace semirings.

Lemma 12.3. *The power-set algebra $2^{P \times P}$ is an i-semiring.*

We call this i-semiring *full relation semiring* over P. It is the homomorphic image of a full trace semiring. Again, by the HSP-theorem, all subalgebras of full relation semirings are i-semirings; complete subalgebras are called *relation semirings*.

Proposition 12.4. *Every identity that holds in all trace semirings holds in all relation semirings.*

We can again take the congruence $\sim_R$, but multiplication is not well-defined in general on equivalence classes and the quotient structures induced are not semirings.

Lemma 12.5. *There is no trace semiring over P and A that is isomorphic to the full relation semiring over a finite set Q with $|Q| > 1$.*

Proof. If there is at least one action in the trace semiring, then the trace semiring is infinite whereas the size of the relation semiring is $2^{|Q|^2}$. Otherwise, all traces will be single states and multiplication will therefore commute on the trace semiring, but not on the relation semiring. Therefore there cannot exist a isomorphism. □

A homomorphism that sends path semirings to relation semirings can be built in the same way as ϕ_R and $\phi_{\overline{R}}$, but using paths instead of a traces as an input. The homomorphism $\chi : 2^{A^*} \to 2^{A^* \times A^*}$ that sends language semirings to relation semirings uses a standard construction (cf. [22]). It is defined, for all $L \subseteq A^*$ by $\tilde{\chi}(L) = \{(v, v.w) : v \in A^* \text{ and } w \in L\}$.

Lemma 12.6. *Every identity that holds in all path or language semirings holds in all relation semirings.*

It is important to distinguish between relation semirings and relational structures under addition and multiplication in general.

Example 12.7. The relational structure from Example 8.2 is *not* a relation semiring, since its greatest element, the set $S = \{(n, n + k) : n, k \in \mathbb{N}\}$, differs from the greatest element $\mathbb{N} \times \mathbb{N}$ of any relation semiring over $\mathbb{N}$. Therefore, by definition, the example semiring is not a subalgebra of any relation semiring.

This fact will explain the deviant behaviour of this semiring in Example 8.2 and in later sections.

13 Star and Omega for Traces, Paths and Languages

In the previous sections we discussed star and omega in finite structures and presented two relational examples. We will now study star and omega in (infinite) trace, path and language semirings. We will relate the results obtained with divergence in Section 15. We will also study omega and divergence on relation semirings in that section.

By definition, trace, path and language semirings are complete and satisfy all necessary infinite distributivity laws. They are $*$-continuous and ω-co-continuous; all fixed points exist and can be determined by iteration.

In all these cases, the calculation of star is straightforward. It gives the reflexive transitive closure, as expected. We therefore focus on the omega.

Lemma 13.1.

(a) *In trace semirings $a^\omega = (a_a)^* a_t \top$ for any $a \in 2^{(P,A)^*}$.*

(b) *In language semirings* $a^\omega = A^*$ *if* $\varepsilon \in a$ *and* $\emptyset$ *otherwise for any* $a \in 2^{A^*}$.
(c) *In path semirings* $a^\omega = a^* a_t \top$ *with* $a_t = a \cap (P \times P)$ *for any* $a \in 2^{P^*}$.

Proof. We first consider trace semirings. Every set of traces S can be partitioned in its *test part* $S_t = S \cap P$ and its *test-free* or action *part* $S_a = S - P$:

$$S = S_t + S_a.$$

This allows us to calculate $(S_t)^\omega$ and $(S_a)^\omega$ separately and then to combine them by Lemma 7.1(d) to $S^\omega = (S_a)^\omega + (S_a)^*(S_t)^\omega$.

On the one hand, $(S_t)^\omega = S_t\top = \{p.\sigma : p \in S \cap P\}$ by Lemma 7.1(e). Informally, this represents the set of all traces starting from some state $p \in S \cap P$.

On the other hand, $(S_a)^\omega = \inf((S_a)^i\top : i \in \mathbb{N}) = \inf((S_a)^i : i \in \mathbb{N})\top$. Since S_a is test-free, every trace $\tau \in S_a$ satisfies $|\tau| > 1$. Therefore, by induction, $|\tau| > n$ for all $\tau \in (S_a)^n$ and consequently $\inf((S_a)^i : i \in \mathbb{N}) = \emptyset$.

As a conclusion, $S^\omega = (S_a)^* S_t \top = S^* S_t \top$ by Lemma 7.1.

By Propositions 10.6 and Proposition 11.5, the argument adapts to language and path semirings. In particular, the test algebras of language algebras are always $\{\emptyset, \{\varepsilon\}\}$. Therefore $L^\omega = 0$ iff $\varepsilon \notin L$ for every language $L \in 2^{A^*}$. □

Lemma 13.1 shows that in trace, path and languages semirings omega can be explicitly defined by the star. This might be surprising: Omega, which seemingly models infinite iteration, reduces to finite iteration after which a miracle (*anything*) happens. Moreover, if an element is test-free, its "infinite iteration" yields zero.

In relation semirings the situation is different: there is no notion of length that would increase through iterations. We will therefore determine omegas in relation semirings relative to a notion of divergence.

14 Divergence Semirings

Divergence can be axiomatised algebraically on i-semirings with additional modal operators. The resulting divergence semirings are similar to Goldblatt's *foundational algebras* [12].

An i-semiring S is called *modal* [21] if it can be endowed with a total operation $\langle a \rangle : \mathsf{test}(S) \to \mathsf{test}(S)$, for each $a \in S$, that satisfies the axioms

$$\langle a \rangle p \leq q \Leftrightarrow ap \leq qa \quad \text{and} \quad \langle ab \rangle p = \langle a \rangle \langle b \rangle p.$$

Intuitively, $\langle a \rangle p$ characterises the set of states with at least one a-successor in p. A *domain* operation $\mathsf{dom} : S \to \mathsf{test}(S)$ is obtained from the diamond operator as $\mathsf{dom}(a) = \langle a \rangle 1$. Alternatively, domain can be axiomatised on i-semirings, even equationally , from which diamonds are defined as $\langle a \rangle p = \mathsf{dom}(ap)$ [9]. The axiomatisation of modal semirings extends to modal Kleene algebras and modal omega algebras without any further modal axioms.

We will use the following properties of diamonds and domain: $\langle p \rangle q = pq$, $\mathsf{dom}(a) = 0 \Leftrightarrow a = 0$, $\mathsf{dom}(\top) = 1$, $\mathsf{dom}(p) = p$. Also, domain is isotone and diamonds are isotone in both arguments.

A modal semiring S is a *divergence semiring* [9] if it can be endowed with a total operation $\nabla : S \to \mathsf{test}(S)$ that satisfies the ∇-unfold and ∇-co-induction axioms

$$\nabla a \leq \langle a \rangle \nabla a \quad \text{and} \quad p \leq \langle a \rangle p \Rightarrow p \leq \nabla a.$$

We call ∇a the *divergence* of a. This axiomatisation can be motivated on trace semirings as follows: The test $p - \langle a \rangle p$ characterises the set of a-maximal elements in p, that is, the set of elements in p from which no further a-action is possible. ∇a therefore has no a-maximal elements by the ∇-unfold axiom and by the ∇-co-induction axiom it is the greatest set with that property. It is easy to see that $\nabla a = 0$ iff a is Noetherian in the usual set-theoretic sense. Divergence therefore comprises the standard notion of program termination. All those states that admit only finite traces are characterised by the complement of ∇a.

It follows from fixed point fusion that the ∇-co-induction axiom is equivalent to

$$p \leq q + \langle a \rangle p \Rightarrow p \leq \nabla a + \langle a^* \rangle q,$$

which has the same structure as the omega co-induction axiom [9]. In particular, ∇a is the greatest fixed point of the function $\lambda x.\langle a \rangle x$, which corresponds to a^ω and $\nabla a + \langle a^* \rangle q$ is the greatest fixed point of the function $\lambda x.q + \langle a \rangle x$, which corresponds to $a^\omega + a^* b$. Moreover, the least fixed point of $\lambda x.q + \langle a \rangle x$ is $\langle a^* \rangle q$, which corresponds to $a^* b$. These fixed points are now defined on test algebras, which are Boolean algebras. Iterative solutions exist again when the test algebra is complete and all diamonds are defined. Then

$$\nabla a = \inf(\langle a^i \rangle 1 : i \in \mathbb{N}) = \inf(\mathsf{dom}(a^i) : i \in \mathbb{N}).$$

However, as the algebra A_3^2 shows, even finite i-semirings, which have a complete test algebra, need not be modal semirings (cf. Example 15.3 below).

We will need the following properties of divergence.

Lemma 14.1. *In every divergence semiring ∇ is isotone and*

$$\langle a \rangle \nabla a \leq \nabla a, \qquad \nabla p = p, \qquad \nabla a \leq \mathsf{dom}(a).$$

Additional properties can be found in [9].

15 Divergence Across Models

We will now relate omega and divergence in all models presented so far. Concretely, we will validate the identities $(\nabla a)\top = a^\omega$ and $\nabla a = \mathsf{dom}(a^\omega)$ that arose from our motivating example in Section 8. We will say that omega is *tame* if every a satisfies the first identity; it will be called *benign* if every a satisfies the second one. We will also be interested in the *taming condition* $\mathsf{dom}(a)\top = a\top$.

First, we consider these properties on relation semirings which we could not treat as special cases of trace semirings in Section 9.

Lemma 15.1. *All relation semirings satisfy the taming condition. Omega is tame and benign.*

Proof. $\mathsf{dom}(a)\top = a\top$ in relation algebras [24], whence in relation semirings.

For tameness, we use the taming condition and infinite distributivity:

$$\begin{aligned}(\nabla a)\top &= \inf(\mathsf{dom}(a^i) : i \in \mathbb{N})\top \\ &= \inf(\mathsf{dom}(a^i)\top : i \in \mathbb{N}) \\ &= \inf(a^i\top : i \in \mathbb{N}) = a^\omega.\end{aligned}$$

A similar proof shows that omega is benign. □

Therefore, omega and divergence are related in relation semirings as expected and, as a special case, $a^\omega = 0$ iff a is Noetherian in relation semirings.

We now revisit the finite i-semirings of Examples 6.5 and 6.6.

Example 15.2. In the Boolean semiring, $\mathsf{dom}(0) = 0$ and $\mathsf{dom}(1) = 1$. Therefore, by Lemma 14.1, $\nabla 0 = 0$ and $\nabla 1 = 1$.

Example 15.3. In A_3^1 and A_3^3, the test algebra is always $\{0, 1\}$; $\mathsf{dom}(0) = 0$ and $\mathsf{dom}(1) = 1$. Moreover, by Lemma 14.1, $\nabla 0 = 0$ and $\nabla 1 = 1$. Setting $\mathsf{dom}(a) = 1 = \nabla a$ turns both into divergence semirings. In contrast, domain cannot be defined on A_3^2.

Consequently, omega is not tame in A_3^2, since $\nabla a\top$ is undefined here, and in A_3^3. However, it is tame in A_3^1 and A_2. In all four finite i-semirings, omega is benign.

Let us now consider trace, path and language semirings. Domain, diamond and divergence can indeed be defined on all these models. On a trace semiring,

$$\mathsf{dom}(S) = \{p : p \in P \text{ and } p.\sigma \in S\}.$$

So, as expected, $\nabla S = \inf(\mathsf{dom}(S^i) : i \in \mathbb{N})$ characterises all states where infinite paths may start. However, since the omega operator is related to finite behaviours in all these models (cf. Lemma 13.1), the expected relationships to divergence fail.

Lemma 15.4. *The taming condition does not hold on some trace and path semirings. Omega is neither tame nor benign.*

Proof. Consider the case of trace semirings. Let $P = \{p\}$ and $A = \{a\}$ and let S be the set consisting of the single trace *pap*. Then $\mathsf{dom}(S) = \{p\} = \nabla S$ and $\mathsf{dom}(S)\top = \{p\}\top = \nabla(S)\top$ is the set of all non-empty traces over p and a. Moreover, $S\top = \{p.a.\tau : \tau \in (P, A)^*\}$. Finally, by Lemma 13.1(a) $S^\omega = (S_a)^* S_t \top = \emptyset$ since $S_t = \emptyset$ in the example. This refutes all three identities for trace semirings. The argument translates to path semirings by forgetting actions. □

The situation for language semirings, where states are forgotten, is different.

Lemma 15.5.

(a) *The taming condition does not hold in some language semirings.*

(b) *Omega is tame in all language semirings.*
(c) $(\nabla a)\top = a^\omega \not\Rightarrow \mathsf{dom}(a)\top = a\top$ *in some language semirings.*

Proof. In language semirings the test algebra is $\{\emptyset, \{\varepsilon\}\}$. So $\mathsf{dom}(L) = \{\varepsilon\}$ iff $L \neq 0$ for every $L \in 2^{A^*}$.

(a) Consider the language semiring over the single letter a and the language $L = \{a\}$. Then $\mathsf{dom}(L) = \{\varepsilon\}$ and therefore $\mathsf{dom}(L)\top = \top \neq L\top$, since $\varepsilon \in \top$, but $\varepsilon \notin L\top$.
(b) $\nabla L = \inf(\mathsf{dom}(L^i) : i \in \mathbb{N}\} = \{\varepsilon\}$ iff $L \neq \emptyset$. Therefore $(\nabla L)\top = \top$ iff $L \neq \emptyset$ and $(\nabla L)\top = \emptyset$ iff $L = \emptyset$. It has already been shown in Lemma 13.1(b) that L^ω satisfies the same conditions.
(c) Immediate from (a) and (b). □

In the next section we will provide an abstract argument that shows that omega is benign on language semirings (without satisfying the taming condition).

Example 15.6. We now compute ∇a for $a = \{(n, n+1 : n \in \mathbb{N}\}$ from Example 8.2. We can iterate

$$\begin{aligned}\nabla a &= \inf(\mathsf{dom}(a^i) : i \in \mathbb{N})\\ &= \inf(\mathsf{dom}(\{(n, n+k) : k \geq i\}) : i \in \mathbb{N})\\ &= \{(n, n) : n \in \mathbb{N}\}.\end{aligned}$$

It has already been shown that $a^\omega = \emptyset$ (cf. Example 8.2). It immediately follows that omega is neither tame nor benign in this structure. It also does not satisfy the taming condition, since

$$\mathsf{dom}(a)\top = \top \neq \{(n, n+k) : k \geq 1\} = a\top.$$

Remember that this relational structure is *not* a relation semiring in Example 12.7. This result therefore does not contradict the statement of Lemma 15.1.

As a conclusion, omega behaves as expected in relation semirings, but not in trace, path and language semirings. This may be surprising: While relations are standard for finite input/output behaviours, traces, paths and languages are standard for infinite behaviours, including reactive and hybrid systems.

16 Taming the Omega

Our previous results certainly deserve a model-independent analysis. We henceforth briefly call *omega divergence semirings* a divergence semiring that is also an omega algebra. We will now consider tameness of omega for this class. It is easy to show that the simple identities

$$a\top \leq \mathsf{dom}(a)\top, \quad a^\omega \leq (\nabla a)\top, \quad \mathsf{dom}(a^\omega) \leq \nabla a,$$

hold in all omega divergence semirings. Therefore we only need to consider the relationships between their converses.

Proposition 16.1. *In the class of omega divergence semirings, the following implications hold, but not their converses.*

$$\mathsf{dom}(a)\top \leq a\top \Rightarrow (\nabla a)\top \leq a^\omega \Rightarrow \nabla a \leq \mathsf{dom}(a^\omega).$$

Proof. For the first implication, $(\nabla a)\top \leq \mathsf{dom}(a(\nabla a)) \leq a(\nabla a)\top$ holds by ∇-unfold and the definition of domain in the first and the assumption in the second step. Then $(\nabla a)\top \leq a^\omega$ by ∇-co-induction.

Its converse fails in the class of language semirings by Lemma 15.5(c).

For the second implication, let $(\nabla a)\top \leq a^\omega$. Then $\nabla a \leq \mathsf{dom}(a^\omega)$ holds by isotonicity of domain and since $\mathsf{dom}(p\top) = \mathsf{dom}(p\ \mathsf{dom}(\top)) = \mathsf{dom}(p1) = p$ for all tests p.

The converse implication fails in A_3^3 since $\nabla a = 1 = \mathsf{dom}(a) = \mathsf{dom}(a^\omega)$, but $(\nabla a)\top = 1 > a = a^\omega$ by Example 6.6 and 15.3. □

Proposition 16.1 shows that the tameness condition implies that omega is tame, which again implies that omega is benign. The fact that omega is benign whenever it satisfies the taming condition has already been proved in [9].

To round up the picture, we will consider the additional condition

$$\mathsf{dom}(a^\omega)\top = a^\omega\top = a^\omega$$

which is similar to the taming condition. Again, it is easy to show that $a^\omega \leq \mathsf{dom}(a^\omega)\top$.

Lemma 16.2. *In the class of omega divergence semirings,*

(a) $(\nabla a)\top = a^\omega \Rightarrow \mathsf{dom}(a^\omega)\top \leq a^\omega$,
(b) $\nabla a \leq \mathsf{dom}(a^\omega) \not\Rightarrow \mathsf{dom}(a^\omega)\top \leq a^\omega$,
(c) $\nabla a \leq \mathsf{dom}(a^\omega) \not\Leftarrow \mathsf{dom}(a^\omega)\top \leq a^\omega$.

Proof. (a) $\mathsf{dom}(a^\omega)\top \leq a^\omega$ is immediate from $\mathsf{dom}(a^\omega) \leq \nabla a$.

(b) In A_3^3, $\nabla a = 1 = \mathsf{dom}(a) = \mathsf{dom}(a^\omega)$ by Example 6.6 and 15.3. However, $\mathsf{dom}(a^\omega)\top = \mathsf{dom}(a)\top = 1 > a = a^\omega$. So $\nabla a \leq \mathsf{dom}(a^\omega) \not\Rightarrow \mathsf{dom}(a^\omega)\top \leq a^\omega$.

(c) By Example 15.6, $\nabla a \neq 0 = a^\omega = a^\omega\top$. It has already been shown that the underlying structure is a omega divergence semiring [9]. □

The remaining relationships with tameness, benignity and the taming conditions are collected in the following corollary. They follow by transitivity.

Corollary 16.3. *In the class of omega divergence semirings,*

(a) $\mathsf{dom}(a)\top = a\top \Rightarrow \nabla a = \mathsf{dom}(a^\omega)$,
(b) $\mathsf{dom}(a)\top = a\top \not\Leftarrow \nabla a = \mathsf{dom}(a^\omega)$,
(c) $\mathsf{dom}(a)\top = a\top \Rightarrow \mathsf{dom}(a^\omega)\top = a^\omega$,
(d) $\mathsf{dom}(a)\top = a\top \not\Leftarrow \mathsf{dom}(a^\omega)\top = a^\omega$,
(f) $\mathsf{dom}(a^\omega)\top = a^\omega \not\Rightarrow (\nabla a)\top = a^\omega$.

All these relationships are depicted in Figure 2.

This concludes our investigation of divergence and omega. It turns out that these two notions of non-termination are unrelated in general. Properties that seem intuitive for relations can be refuted on three-element or natural infinite models. On relation semirings, omega seems consistent with the demonic view on total program correctness. On traces, paths and languages, it vanishes on pure actions that do not contain a test part. The taming condition that seems to play a crucial role could only be verified on (finite and infinite) relation semirings. All possible behaviours arise already for small finite models. Divergence has solid foundations based on set-theoretic intuition. It behaves as expected on all models considered and therefore seems very promising for modelling infinite behaviours.

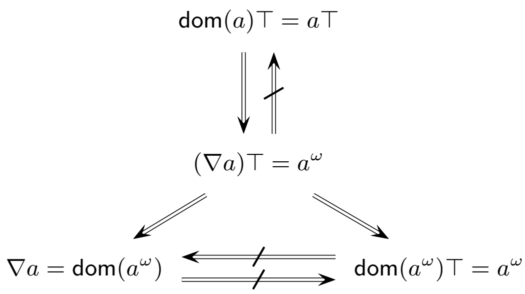

Fig. 2. Relationships between a^ω and ∇a.

17 Conclusion

We compared two notions of non-termination in the context of idempotent semirings: infinite iteration as modelled by the omega operator and divergence as defined on modal semirings. It turned out that divergence models the expected behaviour on standard models such as relations, traces, paths and languages. The omega, however, shows surprising anomalies. In particular, omega is not benign (whence not tame) on traces and paths, which are among the standard models for systems with infinite behaviours such as reactive and hybrid systems.

Our approach considers *infinite* behaviour on *finite* traces, words and paths. Nevertheless, divergence detects the correct infinite behaviour that arises from unravelling labelled transition systems. But omega algebras are by definition not appropriate for infinite behaviour: The right zero axiom $a0 = 0$ excludes that a is an infinite element. In general it seems unreasonable to sequentially compose an infinite element a with another element b to ab. Two alternatives to omega algebras allow adding infinite elements: The weak omega algebras introduced by von Wright [25] and elaborated by Möller [20], and in particular the module-based structures introduced by Ésik and Kuich [11], in which finite and infinite elements have different sorts. It seems very promising to adapt divergence to the module-based setting and to compare the resulting notions with the module-based omega on truly infinite models.

Finally, we do not know whether the identity $\nabla a = \mathsf{dom}(a^\omega)$ holds for all finite i-semirings (the identity $(\nabla a)\top = a^\omega$ fails already on A_3^3). Mace4 shows that there is no counterexample with less than 11 elements; beyond that size, the question remains open. We also do not know whether the axiomatisation of the reflexive transitive closure implies the star induction laws.

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A RelView Code

This section provides the code for calculating star, omega and divergence in finite relation algebras using RelView [2]. The code is also available at `http://www.dcs.shef.ac.uk/~peterh/publications/non-termination/`. The relation a presented in Section 8 can be implemented as follows. The first four lines represent a as a matrix, the last five lines represent it as graph.

```
a (4, 4)
1 : 2, 4
2 : 3
3 : 2

a (4)
{198,348},2,4
{48,197},3
{198,48},2
{347,198}
```

The iteration of star, omega and divergence can be implemented as follows: Note that `divergence` returns a vector and therefore the result cannot be presented in a graph (only as a matrix).

```
{
  star(R) calculates the Kleene star (reflexive transitive closuer)
  Input  R: a homogeneous relation (n x n)
  Output P: a homogeneous relation representing the star (n x n)
}
star(R)
  DECL P
  BEG P =  trans(refl(R))
      RETURN P
  END.

{
  omega(R) calculates the omega operator
  Input  R: a homogeneous relation (n x n)
  Output P: a homogeneous relation representing the omega (n x n)
}
omega(R)
  DECL P,Q
  BEG P = L(R);
      Q = O(R);
      WHILE -eq(P,Q) DO
        Q = P;
        P = R*P
        OD
      RETURN P
  END.

{
  divergence(R) calculates the divergence operator
  Input  R: a homogeneous relation (n x n)
  Output p: a column vector representing the divergence of R (1 x n)
}
divergence(R)
  DECL p,q
  BEG p = Ln1(R);
      q = On1(R);
      WHILE -eq(p,q) DO
        q = p;
        p = dom(R*p)
        OD
      RETURN p
  END.
```

B Mace4 Code

This section provides the code for generating i-semirings and finite omega divergence semirings using Mace4 [1]. The code is again available at our web-page `http://www.dcs.shef.ac.uk/~peterh/publications/non-termination/`. We present the code for generating i-semirings first. Since it contains less axioms than the one for generating omega divergence semirings it is much faster. To generate (all) i-semirings of dimension n call

```
mace4 -n<dim> -f isemiring.in|get_interps|isofilter.
```

Starting it without specifying the dimension produces all i-semirings up to dimension 5.

```
set(print_models_portable).
assign(iterate_up_to,5).
assign(max_models,5000000).

op(500, infix_left, "+").
op(490, infix_left, ";").

formulas(sos).

%standard axioms of i-semirings %%%%%%%%%%%%%%%%%%%%%%%%

  all x all y (x + y = y + x).
  all x (x + 0 = x).
  all x all y all z (x+(y+z) = (x+y)+z).
  all x (x + x = x).

  all x (x;1 = x & 1;x = x).
  all x all y all z (x;(y;z) = (x;y);z).
  all x (0;x = 0).
  all x (x;0 = 0).

  all x all y all z (x;(y + z) = x;z + x;y).
  all x all y all z ((x + y);z = x;z + y;z).

end_of_list.
```

The code for generating omega divergence semirings is similar, but contains much more axioms.

```
%BEGIN
   The Kleene star is presented by ' omega by ~
   The operator * is binary in Prover9/Mace4 by definition
   Redefintion is possible for Prover9 and Mace4, but yields
   problems in add on programs like isofilter
END%

set(print_models_portable).
assign(iterate_up_to,5).
assign(max_models,5000000).

op(500, infix_left, "+").   %Addition
op(490, infix_left, ";").   %Mutiplication
op(480, postfix, "'").      %Kleene star
op(480, postfix, "~").      %Omega
%dom = domain
%div = divergence
%dia = diamond

lex([ 0, 1, ;, +, c, dom, dia, ', ~ ]).
```

```
formulas(sos).

%standard axioms of i-semirings %%%%%%%%%%%%%%%%%%%%%%%%
  all x all y (x + y = y + x).
  all x (x + 0 = x).
  all x all y all z (x+(y+z) = (x+y)+z).
  all x (x + x = x).

  all x (x;1 = x & 1;x = x).
  all x all y all z (x;(y;z) = (x;y);z).
  all x (0;x = 0).
  all x (x;0 = 0).

  all x all y all z (x;(y + z) = x;z + x;y).
  all x all y all z ((x + y);z = x;z + y;z).

% Kleene algebras %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  all x (1 + x ; x' = x').
  all x (1 + x' ; x = x').
  all x all y all z ((x;y + z) + y = y -> x' ; z + y = y).
  all x all y all z ((y;x + z) + y = y -> z ; x' + y = y).

%omega algebras %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  all x (x;x~ = x~ ).
  all x all y all z (y + (x;y + z)= x;y + z -> y + (x~ + x' ; z) = x~ + x' ; z).

% tests %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  test(0).
  test(1).
  all p all q (test(p) & test(q) -> c(p+q) = c(p);c(q) & c(p;q) = c(p)+c(q)).
  all p (test(p) -> p;c(p) = 0 & p+c(p) = 1).
  all p (test(p) -> c(c(p))=p).
  %additional axioms, since in prover9/mace4 c is a total function
  all x (-test(x) -> c(x) = 0).

%domain %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  all x (test(dom(x))).                 % domain is test
  all x all p (test(p) -> dom(p;x) = p;dom(x)).
  all x (dom(x);x = x).
  all x all y (dom(x;y) = dom(x;dom(y))).

%diamonds %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  all x all p (test(p) -> dia(x,p)=dom(x;p)).
  %additional axioms, since in prover9/mace4 fd... is a total function
  all x all y (-test(y) -> dia(x,y) = 0).

%divergence %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
  all x (test(div(x))).                     %div is test
  all x (div(x) = dia(x,div(x))).
  all x all p (p + dia(x,p) = dia(x,p) -> p + div(x) = div(x)).

end_of_list.
```

C Omega Algebras and Divergence Algebra

The following tables show all omega algebras up to dimension 4, as computed with Mace4. Whenever they can be extended to omega divergence semirings, the additional tables are included. As far as the underlying semiring were presented in Conway's book [8], we also add his numbers.

Dimension 2

$A_2 : 0 < 1$ Conway(1.)

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

$*$	
0	1
1	1

ω	
0	0
1	1

$\langle . \rangle$	0	1
0	0	0
1	0	1

dom	
0	0
1	1

∇	
0	0
1	1

Dimension 3

$A_3^1 : 0 < 1 < a$ Conway(2.)

$+$	0	1	a
0	0	1	a
1	1	1	a
a	a	a	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$*$	
0	1
1	1
a	a

ω	
0	0
1	a
a	a

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1

dom	
0	0
1	1
a	1

∇	
0	0
1	1
a	1

$A_3^2 : 0 < a < 1$ Conway(3.)

$+$	0	1	a
0	0	1	a
1	1	1	1
a	a	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

$*$	
0	1
1	1
a	1

ω	
0	0
1	1
a	0

$A_3^3 : 0 < a < 1$ Conway(4.)

$+$	0	1	a
0	0	1	a
1	1	1	1
a	a	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

$*$	
0	1
1	1
a	1

ω	
0	0
1	1
a	a

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1

dom	
0	0
1	1
a	1

∇	
0	0
1	1
a	1

Dimension 4

$A_4^1 : 0 < b < a < 1$ Conway(13.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	0	0

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	0

$A_4^2 : 0 < a < b < 1$ Conway(14.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	0	a

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	0

$A_4^3 0 < a < b < 1$ Conway(15.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	0	b

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	b

$A_4^4 : 0 < a < b < 1$ Conway(18.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	0
b	0	b	a	b

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	b

$A_4^5 : 0 < a < b < 1$ Conway(16.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	a
b	0	b	0	b

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	b

$A_4^6 : 0 < a < b < 1$ Conway(19.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	0	a
b	0	b	a	b

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	0
b	b

$A_4^7 : 0 < a, b < 1$ Conway(17.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	1
b	b	1	1	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

$*$	
0	1
1	1
a	1
b	1

ω	
0	0
1	1
a	a
b	b

$\langle . \rangle$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

dom	
0	0
1	1
a	a
b	b

∇	
0	0
1	1
a	a
b	b

$A_4^8 : 0 < a < b < 1$ Conway(20.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	a

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	1	0	1	1	1
a	1	a	0	1	1	1
b	1	a	0	1	1	1

$A_4^9 : 0 < a < b < 1$ Conway(21.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	b
b	b	1	b	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	b

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	1	0	1	1	1
a	1	a	0	1	1	1
b	1	b	0	1	1	1

$A_4^{10} : 0 < b, 1 < a$ Conway(24.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	a
a	a	a	a	a
b	b	a	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	1

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	a	0	1	1	1
a	a	a	0	1	1	1
b	a	a	0	1	1	1

$A_4^{11} : 0 < b, 1 < a$ Conway(25.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	a
a	a	a	a	a
b	b	a	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	a

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	a	0	1	1	1
a	a	a	0	1	1	1
b	a	a	0	1	1	1

$A_4^{12} : 0 < 1 < b < a$ Conway(6.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	b
a	a	a	a	a
b	b	b	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	a

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	a	0	1	1	1
a	a	a	0	1	1	1
b	a	a	0	1	1	1

$A_4^{13} : 0 < b < 1 < a$ Conway(12.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	1
a	a	a	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	b

	$*$	ω	$\langle . \rangle$ 0	$\langle . \rangle$ 1	dom	∇
0	1	0	0	0	0	0
1	1	a	0	1	1	1
a	a	a	0	1	1	1
b	1	a	0	1	1	1

$A_4^{14} : 0 < 1 < b < a$

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	b
a	a	a	a	a
b	b	b	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	a	b

	$*$	ω	dom	∇
0	1	0	0	0
1	1	a	1	1
a	a	a	1	1
b	b	a	1	1

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1
b	0	1

$A_4^{15} : 0 < b < 1 < a$ Conway(10.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	1
a	a	a	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	a
b	0	b	b	b

	$*$	ω	dom	∇
0	1	0	0	0
1	1	a	1	1
a	a	a	1	1
b	1	b	1	1

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1
b	0	1

$A_4^{16} : 0 < b < 1 < a$ Conway(8.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	1
a	a	a	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	b	0

	$*$	ω
0	1	0
1	1	a
a	a	a
b	1	0

$A_4^{17} : 0 < b, 1 < a$ Conway(22.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	a
a	a	a	a	a
b	b	a	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	b	0

	$*$	ω
0	1	0
1	1	a
a	a	a
b	a	0

$A_4^{18} : 0 < b < 1 < a$ Conway(11.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	1
a	a	a	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	a	b

	$*$	ω	dom	∇
0	1	0	0	0
1	1	a	1	1
a	a	a	1	1
b	1	a	1	1

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1
b	0	1

$A_4^{19} : 0 < b < 1 < a$ Conway(9.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	1
a	a	a	a	a
b	b	1	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	b	b

	$*$	ω	dom	∇
0	1	0	0	0
1	1	a	1	1
a	a	a	1	1
b	1	b	1	1

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1
b	0	1

$A_4^{20} : 0 < b, 1 < a$ Conway(23.)

$+$	0	1	a	b
0	0	1	a	b
1	1	1	a	a
a	a	a	a	a
b	b	a	a	b

$\cdot$	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	b
b	0	b	b	b

$*$	
0	1
1	1
a	a
b	a

ω	
0	0
1	a
a	a
b	b

$\langle . \rangle$	0	1
0	0	0
1	0	1
a	0	1
b	0	1

dom	
0	0
1	1
a	1
b	1

∇	
0	0
1	1
a	1
b	1