Algebraic Notions of Nontermination: Omega and Divergence in Idempotent Semirings

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Abstract

Two notions of nontermination are studied and compared in the setting of idempotent semirings: Cohen's omega operator and a divergence operator. They are determined for various computational models, and conditions for their existence and their coincidence are given. It turns out that divergence yields a simple and natural way of modelling infinite behaviours of programs and discrete systems, whereas the omega operator shows some anomalies.

Key words: idempotent semirings, modal semirings, nontermination, recursion and corecursion, omega, divergence

1. Introduction

Idempotent semirings and Kleene algebras are foundational structures for computing. They provide operations for sequential composition, nondeterministic choice and finite iteration which are essential for computational modelling. They have found applications in program semantics, program analysis and beyond. Since they are based on first-order equational logic, they can be combined with off-the-shelf automated theorem provers into useful verification tools.

Kleene algebras have been extended by omega operations for infinite iteration [3, 23], by domain and modal operators [4, 6, 17] and by operators for program divergence [5, 22]. The so-called omega algebras have been proposed by Cohen for the refinement of concurrent systems [3]. They also found applications in term rewriting [21], and they are related to omega regular languages. A variant is suitable for calculational action system refinement in the demonic setting of the refinement calculus [23].

Among the most fundamental program analysis tasks are termination and nontermination. More generally, modelling equilibrium and nonequilibrium properties via fixed points is central to the analysis of discrete dynamical systems. In a companion paper [5], different algebraic notions of termination based on modal semirings have been introduced and compared. The most important ones are Cohen's omega operator and a divergence operator which captures the standard set-theoretic notion of well-foundedness. Although, intuitively, well-foundedness and absence of infinite iteration, as modelled by the omega operator, should be one and the same concept, it has been shown that they differ on some very natural computational models, including languages.

Here, we extend this investigation to infinity. Our main contributions are as follows.

- We analyse finite and infinite iteration in the semiring setting as fixed points of the mapping $\lambda z.y + xz$ and use some general results about fixed points, (co)recursion and lattices to relate them with Kleene algebras, omega algebras, and fixed point iterations. This yields a fine-grained view of the role of distributivity and (co)continuity in omega algebras and divergence semirings. In particular, Park's observation that infinite iteration on languages is usually not captured by a fixed point [18] generalises to the algebraic setting: in this sense omega does only rarely correspond to tail corecursion and infinite iteration.

- We use some standard techniques from universal algebra and Galois connections for constructing trace, path, language and relation semirings and for relating these structures.

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- We systematically study the omega operator and divergence in various computational models, including finite semirings, relations, traces, paths and languages. It turns out that these two concepts coincide in relation semirings, but they differ on all other models considered, even on some relational models. In particular, we provide abstract arguments showing that omega can be expressed in terms of the Kleene star in many models. This provides further evidence that omega sometimes fails to properly model infinite iteration.
- We also provide abstract *taming conditions* for omega that imply coincidence with divergence. We find a very heterogenous situation: Omega is tame on relation semirings. It is also tame on language semirings, but violates the taming condition. Therefore, the taming condition is only sufficient, but not necessary. In particular, omega is not tame on trace and path semirings.

On the one hand, these results confirm the anomalies of omega found in the companion paper beyond termination. On the other hand, they suggest that divergence semirings [5] are powerful tools that appropriately capture finite and infinite behaviours in some standard computational models for programs and discrete dynamical systems. They provide the right level of abstraction for analysing such systems in a simple and concise way, and for automating this analysis. The omega operator, in contrast, should be applied cautiously and only in certain specific contexts.

The remainder of the paper is organised as follows. Section 2 introduces idempotent semirings and notions of recursion and corecursion on these structures. The Sections 3 to 6 analyse the reductions of (co)recursion to tail (co)recursion and iteration based on fixed point fusion and the Knaster-Tarski theorem. Section 7 specialises these results to finite idempotent semirings and presents some examples. Section 8 presents two examples that show the unexpected behaviour of omega. Sections 9 to 12 define trace, path, language and relation semirings and show some relationships between these models. Section 13 determines finite and infinite iteration in these models, except for relation semirings. Section 14 and 15 formalise a notion of divergence and determine divergences across models. Section 16 presents taming conditions for omega with respect to divergence. Section 17 sums up the results of this paper and discusses some related work as well as some future directions.

We do not display any calculational proofs in this paper. They have all been carried out by the automated theorem prover Prover9 [16]. A template for repeating them can be found in Appendix A. Particular results, and many other theorems about variants of idempotent semirings and Kleene algebras can be found at a web site [9].

2. Idempotent Semirings with (Co)Recursion

Our algebraic analysis of nontermination is based on idempotent semirings. Intuitively, semirings are rings without an additive inverse.

A *semiring* is a structure $(S, +, \cdot, 0, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid, multiplication distributes over addition from the left and right and 0 is a left and right zero of multiplication. A semiring S is *idempotent* (an *i-semiring* or *dioid*) if addition is idempotent, that is, $x + x = x$ holds for all $x \in S$. The concrete axioms can be found — as input for the automated theorem prover Prover9 — in Appendix A.

Idempotent semirings can model actions, programs or state transitions under nondeterministic choice and sequential composition. The element 1 corresponds to the ineffective action — traditionally called `skip` — and the element 0 to the abortive action. We usually omit the multiplication symbol.

Because of idempotence, the additive monoid of an i-semiring forms a semilattice. The i-semiring can therefore be ordered by $x \le y \Leftrightarrow x + y = y$. In fact, 0 is the least element and addition and multiplication are isotone with respect to that order. Another useful concept is *semiring duality*, which holds between statements of a semiring and those of its *opposite* in which the order of multiplication is swapped.

Tests of a program or sets of states of a transition system can also be modelled in this setting. A *test* [14] in an i-semiring S is an element of a Boolean subalgebra $\mathsf{test}(S) \subseteq S$, the *test algebra* of S, such that $\mathsf{test}(S)$ is bounded by 0 and 1 and multiplication coincides with lattice meet. We write $x, y, z \dots$ for arbitrary semiring elements and $p, q, r, \dots$ for tests. Test complementation is denoted by $\neg$. We freely use the laws of Boolean algebras on tests.

Finite and infinite system behaviours can be modelled on i-semirings through fixed points of the endofunctions

$$\mathfrak{f}(X) = y + xX, \qquad \mathfrak{g}(X) = xX \qquad \text{and} \qquad \mathfrak{r}(X) = 1 + xX$$

and their opposites $\hat{\mathfrak{f}}(X) = y + Xx$, $\hat{\mathfrak{g}}(X) = Xx$, and $\hat{\mathfrak{r}}(X) = 1 + Xx$. As usual, the least pre-fixed point and the greatest post-fixed point of a map f are given by

$$\mu_f = \inf(x : f(x) \le x) \qquad \text{and} \qquad \nu_f = \sup(x : x \le f(x)).$$

An $\mathfrak{f}$-*semiring* is an i-semiring on which $\mu_{\mathfrak{f}}$, $\mu_{\hat{\mathfrak{f}}}$, $\nu_{\mathfrak{f}}$ and $\nu_{\hat{\mathfrak{f}}}$ exist for all endofuctions $\mathfrak{f}$. We will show in Lemma 1 that not all i-semirings are $\mathfrak{f}$-semirings.

In general, μ_f is also a least fixed point and ν_f is also a greatest fixed point if f is isotone. We are interested in some special cases and therefore further abbreviate

$$x^* = \mu_{\mathfrak{r}}, \quad x^\omega = \nu_{\mathfrak{g}}, \quad \hat{x}^* = \mu_{\hat{\mathfrak{r}}} \quad \text{and} \quad \hat{x}^\omega = \nu_{\hat{\mathfrak{g}}}.$$

Since addition and multiplication in i-semirings are isotone, the functions $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{r}$ and their opposites are isotone as well. Therefore $\mu_{\mathfrak{f}}$, $\nu_{\mathfrak{f}}$, x^* and x^ω are fixed points. Dual arguments apply to their opposites.

It is obvious that x^* and $\hat{x}^*$ model tail recursions. Since multiplication is noncommutative, x^* and $\hat{x}^*$ can differ. The same argument holds for x^ω and $\hat{x}^\omega$, which model tail corecursions.

3. Kleene Algebras and Omega Algebras

More general forms of tail recursion and tail corecursion on i-semirings have also been considered.

A *Kleene algebra* [13] is an i-semiring that satisfies the *star unfold* and *star induction* axioms

$$1 + xx^* \le x^*, \quad 1 + x^*x \le x^*, \quad y + xz \le z \Rightarrow x^*y \le z, \quad y + zx \le z \Rightarrow yx^* \le z.$$

These axioms imply that x^*y is a least fixed point of $\mathfrak{f}$ and that yx^* is a least fixed point of $\hat{\mathfrak{f}}$. They therefore combine the least fixed point property of $\mathfrak{f}$ with a reduction to tail recursion. The fixed point x^*, however, can still be different from $\mathfrak{r}^*(0)$. Hence tail recursion need not reduce to iteration.

Along similar lines, an *omega algebra* [3] is a Kleene algebra that satisfies the *omega unfold* and the *omega coinduction* axiom

$$x^\omega \le xx^\omega \qquad \text{and} \qquad z \le y + xz \Rightarrow z \le x^\omega + x^*y.$$

Here, the corecursion of $\nu_{\mathfrak{f}}$ splits into a tail recursive and a tail corecursive part that model the finite and the infinite behaviour of $\nu_{\mathfrak{f}}$, respectively. By definition, every omega algebra is an $\mathfrak{f}$-semiring.

Lemma 1.

(*a*) *The fixed point x^*, and therefore $\mu_{\mathfrak{f}}$, need not exist in an i-semiring.*

(*b*) *The fixed point x^ω, and therefore $\nu_{\mathfrak{f}}$, need not exist in a Kleene algebra.*

Proof.

(*a*) The max-plus i-semiring $(\mathbb{N} \cup \{-\infty\}, \max, +, -\infty, 0)$ [8] cannot be extended to a Kleene algebra (cf. [4]).

(*b*) The natural numbers $\mathbb{N}$ with addition and multiplication defined as max, except that $n0 = 0 = 0n$ for all $n \in \mathbb{N}$, form an i-semiring with neutral elements 0 and 1. Multiplication can be approximated by $nm \le \max(n, m)$ for all $m, n \in \mathbb{N}$. Setting $n^* = \max(1, n)$ yields a Kleene algebra:

  (*i*) $1 + nn^* \le n^*$ holds since $1 + nn^* \le \max(1, n, n^*) = \max(1, n) = n^*$.

  (*ii*) For $l + mn \le n \Rightarrow m^*l \le n$, suppose $l + mn \le n$, that is, $\max(l, mn) \le n$. We must show that $\max(l, 1, m) \le n$, which implies that $m^*l \le n$. For $n \ge 1$ the assumption becomes $\max(l, m, n) \le n$, and the claim holds since $\max(l, 1, m) \le \max(l, m, n)$. For $n = 0$ the assumption yields $l = 0$ and the claim becomes $0 \le 0$.

  (*iii*) The dual star unfold and star induction axioms follow from commutativity of multiplication.

$\mathbb{N}$ does not have a greatest element, but 1^ω is the greatest element in every omega algebra (see below). Hence the Kleene algebra just constructed is not an omega algebra. □

Kleene algebras and omega algebras, as opposed to general $\mathfrak{f}$-semirings, are specified within first-order equational logic. Reasoning about the corresponding special forms of recursion and corecursion can therefore be handled by automated theorem provers [10]. Since i-semirings form an equational class, they are, by Birkhoff's HSP-theorem (cf. [24]), closed under subalgebras, direct products and homomorphic images. Since Kleene algebras and omega algebras form universal Horn classes, they are, by Mal'cev's quasi-variety theorem (cf. [24]), closed under subalgebras and direct products, but not necessarily under homomorphic images. These facts are useful for constructing new algebras from given ones. The equational classes generated by Kleene algebras and omega algebras are not finally equationally axiomatisable: Kleene algebras are complete for the equational theory of regular expressions [13] which is not finitely axiomatisable [20]. By this completeness result, all regular identities hold in Kleene algebras and we will freely use them. Examples are $0^* = 1 = 1^*$, $1 \leq x^*$, $xx^* \leq x^*$, $x^*x^* = x^*$, $x \leq x^*$, $x^*x = xx^*$ and $1 + xx^* = x^* = 1 + x^*x$. Furthermore the star is isotone.

It has also been shown that ω-regular identities such as $0^\omega = 0$, $x^\omega = x^\omega 1^\omega$, $x^\omega = xx^\omega$, $x^\omega y \leq x^\omega$, $x^*x^\omega = x^\omega$ and $(x + y)^\omega = (x^*y)^\omega + (x^*y)^*x^\omega$ hold in omega algebras and that omega is isotone. In particular, $x \leq 1^\omega$. Hence each omega algebra has a greatest element which we abbreviate as $\top$. Completeness of omega algebras with respect to ω-regular expressions remains an open problem.

A list of properties of Kleene algebras and omega algebras can again be found at our web site [9]. All of them have been automatically verified. In general, we do not display any calculational proofs in this paper. With state-of-the-art automated theorem proving technology, they can usually be verified easily and quickly. The templates for the automated theorem prover Prover9 and the model generator Mace4 in Appendix A provide the basis for reproducing our proof. Particular results are displayed at our web site.

The following facts help analysing omega in concrete models by splitting actions in a test part and a testfree part.

Lemma 2. *Let x, y, z be elements of some omega algebra and let p be a test.*

(a) $(x + p)x^*p = x^*p$.

(b) $(x + p)^*p = x^*p$.

(c) $(x + y)^\omega = x^\omega + x^*y(x + y)^\omega$.

(d) $(x + p)^\omega = x^\omega + x^*p\top$.

(e) $p^\omega = p\top$.

The proofs are simple exercises in automated deduction.

4. From $\mathfrak{f}$-Semirings to Omega Algebras

Our ultimate goal is to express the fixed points $\mu_\mathfrak{f}$ and $\nu_\mathfrak{f}$ in terms of the tail (co)recursions x^* and x^ω and to link them with iterative solutions. For $\mu_\mathfrak{f}$ and x^* this is well understood (but certainly deserves a reminder). In the context of greatest fixed points and the omega operation on i-semirings, we are not aware of previous investigations.

In this section, we collect some abstract conditions that turn $\mathfrak{f}$-semirings into Kleene algebras and omega algebras. These can be imposed through fixed point fusion theorems (cf. [1]). At the moment we only need their trivial parts.

Lemma 3. *Let f, g and h be functions on some poset; let h be isotone. Then*

(a) *$f \circ h \leq h \circ g \Rightarrow \mu_f \leq h(\mu_g)$, whenever μ_f and μ_g exist,*

(b) *$f \circ h \geq h \circ g \Rightarrow \nu_f \geq h(\nu_g)$, whenever ν_f and ν_g exist.*

In Lemma 3, the order on functions is the pointwise extension of the poset order. The nontrivial part of fixed point fusion deals with the converse inequalities.

Lemma 4. *In every $\mathfrak{f}$-semiring,*

(a) *$\mu_\mathfrak{f} \leq x^*y$ and, dually, $\hat{\mu}_\mathfrak{f} \leq y\hat{x}^*$,* (*b*) $x^\omega + \mu_\mathfrak{f} \leq \nu_\mathfrak{f}$.

Proof. This is a simple application of fixed point fusion. We only show $\mu_\mathfrak{f} \leq x^* y$. If we set $f(X) = \mathfrak{f}(X) = y + xX$, $g(X) = \mathfrak{x}(X) = 1 + xX$ and $h(X) = Xy$ in Lemma 3(a), the claim holds if h is isotone and $f \circ h \leq h \circ g$. First, h is isotone since addition and multiplication are isotone. Second, $(f \circ h)(X) = y + x(Xy) = (1 + xX)y = (h \circ g)(X)$. □

$\mathfrak{f}$-semirings become Kleene algebras if the converse inequalities to those in Lemma 4(a) hold. To enforce also $x^* = \hat{x}^*$, Leiß [15] has proposed the conditions

$$yx^* \leq \hat{\mu}_\mathfrak{f} \quad \text{and} \quad \hat{x}^* y \leq \mu_\mathfrak{f}.$$

By Lemma 4 these *Leiß conditions* imply that $x^* y = \mu_\mathfrak{f}$, $y\hat{x}^* = \hat{\mu}_\mathfrak{f}$ as well as $x^* = \hat{x}^*$ hold in $\mathfrak{f}$-semirings. Similarly, for omega algebras, $\nu_\mathfrak{f} = x^\omega + \mu_\mathfrak{f}$ can be enforced by the condition

$$\nu_\mathfrak{f} \leq x^\omega + \mu_\mathfrak{f}. \tag{1}$$

The following fact is then straightforward.

Lemma 5.

(*a*) *An $\mathfrak{f}$-semiring is a Kleene algebra if the Leiß conditions hold.*

(*b*) *A Kleene algebra is an omega algebra if condition (1) holds.*

The converse direction of (b) need not hold since $\nu_\mathfrak{f}$ need not exist. The next lemma is useful for computing stars. It has been verified by automated deduction.

Lemma 6. *In every Kleene algebra, x^* formally models the reflexive transitive closure of x. It satisfies $1+x+x^*x^* \leq x^*$ and $1 + x + yy \leq y \Rightarrow x^* \leq y$.*

It seems that Lemma 6 does not hold in arbitrary $\mathfrak{f}$-semirings. But we do not have a counterexample. We also do not know whether the reflexive transitive closure laws imply the star induction axioms in i-semirings. But we will later present sufficient conditions for this implication.

5. Continuity and Finite Iteration

We now link the Leiß conditions and condition (1) with more concrete concepts, such as continuity and fixed point iteration. We first recall some basic facts. It is easy to show that every isotone function f on a semilattice with zero satisfies $f^n(0) \leq \mu_f$, whenever μ_f exists. The definitions

$$f^{(n)}(x) = \sup(f^i(x) : 0 \leq i \leq n) \qquad \text{and} \qquad f^*(x) = \sup(f^i(x) : i \in \mathbb{N}),$$

imply that also

$$f^{(n)}(0) \leq \mu_f \qquad \text{and} \qquad f^*(0) \leq \mu_f$$

whenever the suprema $f^*(0)$ and μ_f exist. In particular, these properties hold for $\mathfrak{f}$ and $\mathfrak{x}$, which are isotone.

But it can be shown that $\mu_\mathfrak{f}$ and x^* need not be equal to the corresponding iterations. Moreover, $\mathfrak{f}^{(n)}(0) = \mathfrak{x}^{(n)}(0)y$ for each $n \in \mathbb{N}$. In fact, equality of $\mathfrak{f}^*(0)$ and $\mathfrak{x}^*(0)y$, that is, reduction of $\mu_\mathfrak{f}$ to tail recursion, requires a continuity property, the infinite distributivity law

$$\sup(x^n y : n \in \mathbb{N}) = \sup(x^n : n \in \mathbb{N})y.$$

Likewise x^* and $\hat{x}^*$ may still be different.

Analogous equations for $\nu_\mathfrak{f}$ and x^ω cannot even be written down in the semiring setting. There is no meet operation for modelling top-down iteration.

Leiß conditions for $\mu_\mathfrak{f}$ can be enforced via the nontrivial direction of fixed point fusion. Here we present a version for our specific applications. A more general statement can be based on Galois connections [1].

Theorem 7. *Let f, g and h be functions on some poset with least element* 0*. Let also $\mu_g = g^*(0)$, let h distribute over arbitrary suprema of $g^n(0)$, and let f be isotone. Then*

$$f \circ h \geq h \circ g \Rightarrow \mu_f \geq h(\mu_g).$$

The proof is a simple calculation. This fusion theorem immediately links $\mu_{\mathfrak{f}}$ with tail recursion. The infinite distributivity law

$$\sup(\mathfrak{x}^n(0) : n \in \mathbb{N})y = \sup(\mathfrak{x}^n(0)y : n \in \mathbb{N}) \tag{2}$$

therefore implies the second Leiß condition $\hat{x}^* y \leq \mu_{\mathfrak{f}}$, and its opposite implies the first Leiß condition. Both can be combined into the $*$-*continuity* axiom

$$xy^*z = \sup(xy^nz : n \in \mathbb{N}).$$

A $*$-*continuous Kleene algebra* [12] is an i-semiring expanded by a star operation that satisfies the $*$-continuity axiom. Lemma 5 and fixed point fusion therefore imply the well-known fact that every $*$-continuous Kleene algebra is a Kleene algebra. On $*$-continuous Kleene algebras, $\mu_{\mathfrak{f}}$ and $\hat{\mu}_{\mathfrak{f}}$ can be computed iteratively as

$$\mu_{\mathfrak{f}} = x^*y = \mathfrak{x}^*(0)\, y \quad \text{and} \quad \hat{\mu}_{\mathfrak{f}} = yx^* = y\, \mathfrak{x}^*(0).$$

The assumptions of the fixed point fusion theorem can also be enforced by requiring that the semilattice reduct of the i-semiring be complete in the sense that arbitrary suprema exist. Then $x^* = \mathfrak{x}^*(0) = \sup(x^i : i \in \mathbb{N})$ follows from the Knaster-Tarski theorem and the assumption that x (left) distributes over all suprema. For $\mu_{\mathfrak{f}} = \mathfrak{x}(0)y = x^*y$, the above infinite distributivity law for y is still required.

The infinite distributivity laws hold a fortiori, for instance, when the i-semiring is defined over a complete Boolean algebra instead of a semilattice. In this case, $\lambda z.xz$ and $\lambda z.zx$ are lower adjoints of the Galois connections defining residuals. They therefore distribute with arbitrary suprema. Some important computational models such as trace, path language and relation semirings, which we consider below, have complete Boolean algebras as reducts.

Finally, in the complete case, the iterative definition of x^* as the reflexive transitive closure of x links x^* and $\mu_{\mathfrak{f}}$ by tail recursion and subsumes the Kleene algebra axioms. This is interesting for computing stars in concrete models.

Lemma 8. *In the class of complete i-semirings with infinite distributivity law (2), the star unfold and star induction axioms hold. They are equivalent to the reflexive-transitive closure axioms from Lemma 6.*

6. Omega, Cocontinuity, and the Failure of Iteration

Can the arguments of the previous section be dualised — in the sense of lattice theory — from the star to the omega? This would require to underpin condition (1) by a dual version of fixed point fusion for greatest fixed points and a cocontinuity argument. The dual fusion theorem can readily be obtained from Theorem 7.

Theorem 9. *Let f, g and h be functions on some poset with greatest element $\top$. Let $\nu_g = g^\omega(\top) = \inf(g^n(\top) : n \in \mathbb{N})$, h be distribute over arbitrary infima of $g^n(\top)$ and f be isotone. Then*

$$f \circ h \leq h \circ g \Rightarrow \nu_f \leq h(\nu_g).$$

Let us analyse the conditions of Theorem 9 for $\mathfrak{f}$-semirings. First, the existence of a greatest element $\top$ is guaranteed in this case. The greatest fixed point property immediately implies that $1^\omega = \top$. Second, instantiating the functions f, g and h suggests a cocontinuity law of the form

$$x^\omega + x^*y = \inf(x^n\top + x^*y : n \in \mathbb{N}).$$

If this law holds and if $x^\omega = \inf(\mathfrak{g}^n(\top) : n \in \mathbb{N})$, then the axioms of omega algebras would indeed correctly represent corecursion on $\mathfrak{f}$-semirings. But can the assumption on x^ω be justified? By the Knaster-Tarski theorem, this would require that multiplication with x distributes over arbitrary infima. Even if the semilattice reduct of the i-semiring is complete — in which case the complete semilattice is also a complete lattice and hence has all infima required — this need not be the case. In fact, Park has presented a counterexample in the semiring of formal languages [18]. This counterexample can easily be generalised to other computationally important models such as traces and paths (cf. Sections 9 to 12). Hence the conditions for applying Theorem 9 are rarely satisfied and the omega algebra axioms cannot in general be justified along the lines of fixed point fusion and the Knaster-Tarski theorem. Normally, we can only expect that

$$x^\omega \leq \inf(x^n\top : n \in \mathbb{N}).$$

Fixed point fusion is often presented in a more general setting where iterative solutions are not needed and the condition of distributivity with respect to arbitrary infima is replaced by that of being an adjoint of a Galois connection (cf. [1]) .

Theorem 10. *Let f, g and h be endofunctions on a complete lattice. Let h be the upper adjoint of a Galois connection. Then*

$$f \circ h \leq h \circ g \Rightarrow \nu_f \leq h(\nu_g).$$

Every upper adjoint is automatically isotone, so that the converse inequalities also hold. On an i-semiring, the map $h = \lambda x.x + \mu_\mathfrak{f}$ can of course not be expected to be an upper adjoint. But again, if the semiring is defined over a complete Boolean algebra, then h is an upper adjoint with respect to complementation. This covers relation, trace, path and language models, as we will see below. More abstract conditions are left for future investigation.

It is well known that on complete lattices, continuity conditions and the existence of adjoints are connected: every map that distributes with all suprema has an upper adjoint; every map that distributes with all infima has a lower one.

In conclusion, our results so far show that, despite the fact that the star and the omega operation arise as fixed points of the same function, the results for the star are not inherited by the omega by duality. The reason is that distributivity of multiplication over suprema is a rather natural condition, whereas distributivity of multiplication over infima can only rarely be assumed in computational models. The standard techniques such as fixed point fusion and the Knaster-Tarski theorem therefore fail to establish a connection between the omega algebra axioms, tail corecursion and infinite iteration. As further evidence, we investigate the omega operation on some computationally important models below. In contrast, we will see that the alternative notion of divergence satisfies the conditions of Theorem 10. It therefore displays a more satisfactory relationship between corecursion and tail corecursion.

7. Finite Idempotent Semirings

Finite i-semirings enjoy all the desirable properties of the previous sections. First, their semilattice reducts are a fortiori complete — even complete lattices — and therefore all relevant fixed points exist. Second, they trivially satisfy all necessary distributivity laws for the star. Therefore, by the Knaster-Tarski theorem, x^* and $\hat{x}^*$ can be computed iteratively as $\mathfrak{x}^*(0)$ and also μ_f can be determined by iteration. Third, in this case, also x^ω can be obtained this way: $x^\omega = \mathfrak{g}^\omega(\top)$, but not because of the Knaster-Tarski theorem. On a finite i-semiring, the fixed point iteration from $\top$ will always become stationary after finitely many steps. Since x^ω is a lower bound to all such iterations it coincides with their limit. Thus, since the cocontinuity law also holds in this setting, $\nu_\mathfrak{f} = x^\omega + \mu_\mathfrak{f}$. This discussion is summed up by the following lemma.

Lemma 11. *Every finite $\mathfrak{f}$-semiring is an omega algebra.*

While the computations of x^* and $\mu_\mathfrak{f}$ are immediate from the addition and multiplication tables, those of x^ω and $\nu_\mathfrak{f}$ require the computations of meets in the semiring. They can be obtained from the *transitive reduct* of $\leq$, which is a least relation r with transitive closure $\leq$. Then $\inf(x, y) = r^\circ(x) \sqcap r^\circ(y)$, where r° denotes the converse of r, $\sqcap$ the meet operation, and $r(x)$ the relational image of x under r.

We have explicitly computed the stars and omegas for a large number of small finite models with the model generator Mace4 [16]. We further analyse these models in Section 15 and use them as counterexamples in Section 16.

Example 12. The two-element Boolean algebra A_2 is an omega algebra with $0^* = 1^* = 1^\omega = 1$ and $0^\omega = 0$. It is the only two-element omega algebra.

Example 13. There are three i-semirings with three elements. Let $\{0, a, 1\}$ be the set of these elements. Only a is free in the defining tables. Stars and omegas are fixed by $0^* = 1^* = 1$, $0^\omega = 0$ and $1^\omega = \top$ (the greatest element) except for a.

(a) In A_3^1, addition is defined by $0 < 1 < a$, moreover, $aa = a^* = a^\omega = a$.

(b) In A_3^2, $0 < a < 1$, $aa = a^\omega = 0$ and $a^* = 1$.

(c) In A_3^3, $0 < a < 1$, $aa = a^\omega = a$ and $a^* = 1$.

A list containing all omega algebras with at most six elements can be found at a web site [9].

8. Omega in Two Examples

This section provides two examples that emphasise the anomalous and heterogenous behaviour of the omega operator on concrete models. We first illustrate the computations of star and omega in a simple finite relational example.

Example 14. Consider the binary relation $x = \{(p,q),(q,r),(r,q),(p,s)\}$ over $P = \{p,q,r,s\}$ depicted in the first graph in Figure 1. The greatest element in the set of all binary relations over P is $\top = P \times P$ and the least element is $\emptyset$.

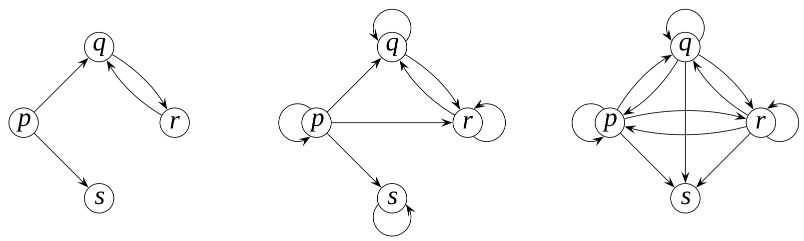

Figure 1: The relations x, x^* and x^ω.

By Lemma 6, x^* is the reflexive transitive closure of x. Hence x^* represents the finite x-paths as follows: $(p,q) \in x^*$ iff there is a finite x-path from p to q. Iterating $x^* = \sup(x^n : n \in \mathbb{N})$ yields

$$x^* = \{(p,p),(p,q),(p,r),(p,s),(q,q),(q,r),(r,r),(r,q),(s,s)\}.$$

This is shown in the second graph in Figure 1.

But what about x^ω? One might expect that this relation represents infinite x-paths in the sense that $(p,q) \in x^\omega$ iff p and q lie on an infinite x-path. But iterating $x^\omega = \inf(x^n\top : n \in \mathbb{N})$ yields

$$x^\omega = \{(p,p),(p,q),(p,r),(p,s),(q,p),(q,q),(q,r),(q,s),(r,p),(r,q),(r,r),(r,s)\},$$

which is shown in the third graph in Figure 1. Note that $(q,p) \in x^\omega$ although there is no x-path from q to p, neither finite nor infinite.

So what does x^ω represent? Let ∇x model those nodes — in the sense of tests — from which x diverges, that is, from which an infinite x-path emanates. Then Example 14 shows that elements in ∇x are linked by x^ω to any other node; elements outside of ∇x are not in the domain of x^ω. Interpreting x^ω generally as *anything for states on which x diverges* would be consistent with the demonic semantics of total program correctness; its interpretation of *nothing for states on which x diverges* models partial correctness. This further suggests to investigate the properties

$$(\nabla x)\top = x^\omega \quad \text{and} \quad \nabla x = d(x^\omega),$$

where $d(x^\omega)$ denotes the domain of the relation x^ω, that is the set of all p with $(p,q) \in x^\omega$. These two identities do not only hold in Example 14; they are of central interest in the following sections. To study them further, we introduce some important models of i-semirings and omega algebras. Then we axiomatise divergence in this setting.

We now revisit a second example that has already been used to show that termination modelled as $x^\omega = 0$ differs from the standard set-theoretic notion of well-foundedness [5].

Example 15. Consider the set $S = \{(n, n+k) : n, k \in \mathbb{N}\}$ of all pairs of natural numbers where the first number is not greater than the second one. It has been shown that 2^S forms a complete Kleene algebra under the usual relational operations of union, relative product and reflexive-transitive closure, and with greatest element S, least element $\emptyset$, and the identity relation as unit. Since the semilattice reduct is complete, star and omega exist and the star can be computed by iteration. Consider now the relation $x = \{(n, n+1) : n \in \mathbb{N}\} \in 2^S$. Then

$$x^\omega \leq \inf(x^i S : i \in \mathbb{N}) = \inf(\{(n, n+k) : k \geq i\} : i \in \mathbb{N}) = \emptyset,$$

since no pair $(n, n+j)$, for arbitrary j, will survive iteration $j+1$ and contribute to the infimum (Section 13 presents an abstract condition to determine when x^ω vanishes). The main issue is that no element in S can reduce the right-hand side of the relation. Nevertheless, x is not well-founded (or, more precisely, *Noetherian*) in the standard set-theoretic sense, since infinite chains $n, n+1, n+2, \dots$ start at each element n. According to the previous discussion, ∇x should therefore be non-empty.

Which conclusion should be drawn from these two examples? Example 14 suggests that x^ω models *anything* for states on which it diverges, whereas Example 15 shows that it models *nothing* for states on which it diverges. So it seems that omega behaves rather erratically. We therefore thoroughly investigate this operation on computationally relevant models in the next few sections.

9. Trace Semirings

In the next four sections we introduce some of the most important models of i-semirings: traces, paths, languages and relations. We then study omega and divergence on these models.

As usual, a *word* over a set Σ is a finite sequence of zero or more elements from Σ. The empty word — the unique sequence of length 0 — is denoted by ε and *concatenation* of words σ_0 and σ_1 by $\sigma_0.\sigma_1$. We write $\mathsf{first}(\sigma)$ for the first element of a word σ and $\mathsf{last}(\sigma)$ for its last element. We write $|\sigma|$ for the length of σ. The set of all words over Σ is denoted by Σ^*.

A (finite) *trace* over the sets P and A is either ε or a word σ such that $\mathsf{first}(\sigma), \mathsf{last}(\sigma) \in P$ and in which elements from P and A alternate. We will use $\tau_0, \tau_1, \dots$ for denoting traces. The product of traces τ_0 and τ_1 is the trace

$$\tau_0 \cdot \tau_1 = \begin{cases} \sigma_0.p.\sigma_1 & \text{if } \tau_0 = \sigma_0.p \text{ and } \tau_1 = p.\sigma_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Intuitively, $\tau_0 \cdot \tau_1$ glues two traces together when the last state of τ_0 and the first state of τ_1 are equal. It then follows that $\mathsf{first}(\tau_0 \cdot \tau_1) = \mathsf{first}(\tau_0)$ and $\mathsf{last}(\tau_0 \cdot \tau_1) = \mathsf{last}(\tau_1)$ whenever this product exists. The set of all traces over P and A is denoted by $(P, A)^*$. Traces naturally arise in the context of labelled transition systems [2] and as an abstract interpretation for program schemes [11]. In that context, P models the state space of the system and A the set of actions it can perform.

Lemma 16. *The powerset algebra $2^{(P,A)^*}$ with addition defined by set union, multiplication by the complex product*

$$T_0 \cdot T_1 = \{\tau_0 \cdot \tau_1 : \tau_0 \in T_0, \tau_1 \in T_1 \text{ and } \tau_0 \cdot \tau_1 \text{ defined}\}$$

P as unit and $\emptyset$ as zero is an i-semiring.

We call this i-semiring the *full trace semiring* over P and A. By definition, $T_0 \cdot T_1 = \emptyset$ if all products between traces in T_0 and traces in T_1 are undefined. Full trace semirings admit rich test algebras: 2^P, for instance, is a complete Boolean algebra by definition.

Every subalgebra of the full trace semiring is, by the HSP-theorem, again an i-semiring. All constants such as 0, 1 and $\top$ are fixed by the subalgebra construction. We will henceforth consider only complete subalgebras of full trace semirings and call them *trace semirings*. Every non-complete subalgebra of the full trace semiring can of course uniquely be closed to a complete subalgebra.

10. Path Semirings

Let us forget all actions of traces. Consider the *projection* $\phi_P : (P, A)^* \to P^*$ defined, for all $p \in P$ and $a \in A$ by

$$\phi_P(\varepsilon) = \varepsilon, \qquad \phi_P(p.\sigma) = p.\phi_P(\sigma), \qquad \phi_P(a.\sigma) = \phi_P(\sigma).$$

ϕ_P is a mapping from traces to words over P which we call *paths*. A path product can be defined as for traces. For paths π_0 and π_1,

$$\pi_0 \cdot \pi_1 = \begin{cases} \sigma_0.p.\sigma_1 & \text{if } \pi_0 = \sigma_0.p \text{ and } \pi_1 = p.\sigma_1, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Again, $\pi_0 \cdot \pi_1$ glues two paths together when the last state of π_0 and the first state of π_1 are equal.

The mapping ϕ_P can be extended to a set-valued mapping $\phi_P : 2^{(P,A)^*} \to 2^{P^*}$ by taking the image:

$$\phi_P(T) = \{\phi_P(\tau) : \tau \in T\}.$$

Now, ϕ_P maps sets of traces to sets of paths.

The information about actions can be introduced to paths by *fibration* with the relational inverse $\phi_P^{-1} : P^* \to 2^{(P,A)^*}$ of ϕ_P. Intuitively, it fills the spaces between states in a path with all possible actions and therefore maps a single path to a set of traces. The mapping ϕ_P^{-1} can again be lifted to the set-valued mapping

$$\phi_P^\sharp(Q) = \sup(\phi_P^{-1}(\pi) : \pi \in Q),$$

where $Q \in 2^{P^*}$ is a set of paths.

Lemma 17. *ϕ_P and $\phi_P^\sharp$ are adjoints of a Galois connection, i.e.,*

$$\phi_P(x) \leq y \Leftrightarrow x \leq \phi_P^\sharp(y).$$

The proof is straightforward. Galois connections give theorems for free. In particular, ϕ_P commutes with all existing suprema and $\phi_P^\sharp$ commutes with all existing infima. Both ϕ_P and $\phi_P^\sharp$ are isotone and the cancellation laws $\phi_P \circ \phi_P^\sharp \leq id_{2^{P^*}}$ and $id_{2^{(P,A)^*}} \leq \phi_P^\sharp \circ \phi_P$ hold. Finally, the mappings are pseudo-inverses: $\phi_P \circ \phi_P^\sharp \circ \phi_P = \phi_P$ and $\phi_P^\sharp \circ \phi_P \circ \phi_P^\sharp = \phi_P^\sharp$.

Lemma 18. *The projections ϕ_P are homomorphisms.*

Proof. We first consider $\phi_P : (P, A)^* \to P^*$. Then $\phi_P(\tau_0 \cdot \tau_1) = \phi_P(\tau_0) \cdot \phi_P(\tau_1)$ and $\phi_P(\varepsilon) = \varepsilon$ are immediate from the definition of trace and path products.

Therefore $\phi_P(T_0 \cdot T_1) = \phi_P(T_0) \cdot \phi_P(T_1)$ as well for sets of traces T_0 and T_1. Moreover, $\phi_P(T_0 + T_1) = \phi_P(T_0) + \phi_P(T_1)$ and $\phi_P(\emptyset) = \emptyset$ follow from the Galois connection. Finally, $\phi_P(P) = P$ holds by definition. □

By the HSP-theorem the set-valued homomorphism induces path semirings from trace semirings.

Lemma 19. *The powerset algebra 2^{P^*} is an i-semiring.*

We call this i-semiring the *full path semiring* over P. It is the homomorphic image of a full trace semiring. Again, by the HSP-theorem, all subalgebras of full path semirings are i-semirings; complete subalgebras are called *path semirings*.

Lemma 20. *Every identity that holds in all trace semirings holds in all path semirings.*

Moreover, the class of trace semirings contains isomorphic copies of all path semirings. This can be seen as follows:

Consider the congruence $\sim_P$ on a trace semiring over P and A that is induced by the homomorphism ϕ_P. The associated equivalence class $[T]_P$ contains all those sets of traces that differ in actions, but not as paths. From each equivalence class we can choose a special canonical representative, which is a set of traces that are built from one single action. Each representative is of course equivalent to a set of paths and therefore an element of a path semiring. Conversely, every element of a path semiring can be expanded to an element of some trace semiring by filling in the same action between all states.

The following lemma can be proved using standard techniques from universal algebra.

Lemma 21. *Let S be the full trace semiring over P and A. The quotient algebra $S/{\sim_P}$ is isomorphic to each full trace semiring over P and $\{a\}$ with $a \in A$ and to the full path semiring over P:*

$$S/{\sim_P} \cong 2^{(P,\{a\})^*} \cong 2^{P^*}.$$

In particular, ϕ_P and $\phi_P^\sharp$ are inverse isomorphisms between the full trace semiring $2^{(P,\{a\})^*}$ and the full path semiring 2^{P^*}, that is, $\phi_P^{-1} = \phi_P^\sharp$.

Lemma 21 is not restricted to full trace and path semirings. It immediately extends to trace and path semirings, since the operations of forming subalgebras and of taking homomorphic images always commute. In particular, each path semiring is isomorphic to some trace semiring with a single action. This isomorphic embedding of path semirings into the class of trace semirings implies the following proposition.

Proposition 22. *Every first-order property that holds in all trace semirings holds in all path semirings.*

11. Language Semirings

Instead of forgetting actions we now forget all states of traces. Most of the arguments of the last section remain valid, but not all of them.

Consider the projection $\phi_L : (P, A)^* \to A^*$ defined, for all $p \in P$ and $a \in A$, by

$$\phi_L(\varepsilon) = \varepsilon, \qquad \phi_L(p.\sigma) = \phi_L(\sigma), \qquad \phi_L(a.\sigma) = a.\phi_L(\sigma).$$

Now ϕ_L maps traces to *words* over A. By definition (cf. Section 9), the product on words is a total function, whereas that on traces and paths is partial. The mapping ϕ_L can again be extended to a set-valued mapping $\phi_L : 2^{(P,A)^*} \to 2^{A^*}$ by taking the image: $\phi_L(T) = \{\phi_L(\tau) : \tau \in T\}$. This extension sends sets of traces to *languages*. Information about states can once more be introduced to words by fibration with the relational inverse $\phi_L^{-1} : A^* \to 2^{(P,A)^*}$ of ϕ_L. Intuitively, all spaces before and after actions in a word are filled with all possible states; a single word is mapped to a set of traces. ϕ_L^{-1} can also be lifted to a set-valued mapping $\phi_L^\sharp(L) = \sup(\phi_L^{-1}(w) : w \in L)$, for any language $L \in 2^{L^*}$.

Lemma 23. *ϕ_L and $\phi_L^\sharp$ are adjoints of a Galois connection.*

Therefore, ϕ_L and $\phi_L^\sharp$ share the properties of ϕ_P and $\phi_P^\sharp$ from the previous section. However, both mappings ϕ_L do not preserve multiplication.

Lemma 24. *The projections ϕ_L need not be homomorphisms.*

Proof. The product $\tau_0 \cdot \tau_1$ is for $\tau_0 = pap$ and $\tau_1 = qap$ is undefined. Thus $\phi_L(\tau_0 \cdot \tau_1)$ is undefined as well, but $\phi_L(\tau_0).\phi_L(\tau_1) = a.a$. This extends to the set-valued case by taking $T_0 = \{pap\}$ and $T_1 = \{qap\}$. □

Note that $\phi_L(T_0 \cdot T_1) \subseteq \phi_L(T_0).\phi_L(T_1)$, but the converse inclusion need not hold.

Here, we cannot use the HSP-theorem together with the set-valued homomorphism to obtain language semirings from trace semirings. Nevertheless the following fact is well known.

Lemma 25. *The powerset algebra 2^{A^*} is an i-semiring.*

We call this i-semiring the *full language semiring* over A. Again, by the HSP-theorem, all its subalgebras are i-semirings; complete subalgebras are called *language semirings*.

Still, the class of trace semirings contains isomorphic copies of all language semirings.

Lemma 26. *Let S be the full trace semiring over $\{p\}$ and A. Then S is isomorphic to the full language semiring over A:*

$$2^{(\{p\},A)^*} \cong 2^{A^*}.$$

We could still define an equivalence relation $\equiv_L$ by partitioning the class of trace semirings according to sets of traces that differ only on states. However, it can be shown along the lines of the proof of Lemma 24 that this equivalence need not be a congruence and therefore the quotient structure is not always a semiring. At least, the mappings ϕ_P and $\phi_P^\sharp$ are isomorphisms between the full trace semiring $2^{(\{p\},A)^*}$ and the full path semiring 2^{A^*}.

Lemma 26 can again be extended to (non-full) trace and language semirings; each language semiring is isomorphic to some trace semiring with one single state. This isomorphic embedding of language semirings into the class of trace semirings implies the following proposition.

Proposition 27. *Every first-order property that holds in all trace semirings holds in all language semirings.*

12. Relation Semirings

Now we forget entire paths between the first and the last state of a trace. We therefore consider the mapping $\phi_R : (P, A)^* \to P \times P$ defined by

$$\phi_R(\tau) = \begin{cases} (\mathsf{first}(\tau), \mathsf{last}(\tau)) & \text{if } \tau \neq \varepsilon, \\ \text{undefined} & \text{if } \tau = \varepsilon. \end{cases}$$

It sends trace products to (standard) relational products on pairs. As before, ϕ_R can be extended to a set-valued mapping $\phi_R : 2^{(P,A)^*} \to 2^{P\times P}$ by taking the image, i.e., $\phi_R(T) = \{\phi_R(\tau) : \tau \in T,\ \phi_R(\tau) \text{ defined}\}$. Now, ϕ_R sends sets of traces to *relations*. Information about the traces between starting and ending state can be introduced to pairs of states by the fibration $\phi_R^{-1} : P \times P \to 2^{(P,A)^*}$ of ϕ_R. Intuitively, it replaces a pair of states by all possible traces between them. It can again be lifted to the set-valued mapping $\phi_R^\sharp(R) = \sup(\phi_R^{-1}(r) : r \in R)$, for any relation $R \subseteq P \times P$.

Lemma 28. *ϕ_R and $\phi_R^\sharp$ are adjoints of a Galois connection.*

Lemma 29. *The projections ϕ_R are homomorphisms.*

By the HSP-theorem the set-valued homomorphism induces relation semirings from trace semirings.

Lemma 30. *The powerset algebra $2^{P\times P}$ is an i-semiring.*

We call this algebra the *full relation semiring* over P. It is the homomorphic image of a full trace semiring. All subalgebras of full relation semirings are again i-semirings; complete subalgebras are called *relation semirings*.

Proposition 31. *Every identity that holds in all trace semirings holds in all relation semirings.*

We can again take the congruence $\sim_R$, but multiplication need not be well defined on equivalence classes and the quotient structures induced are not always semirings.

Lemma 32. *There is no trace semiring over P and A that is isomorphic to the full relation semiring over a finite set Q with $|Q| > 1$.*

Proof. If there is at least one action in the trace semiring, then the trace semiring is infinite whereas the relation semiring has size $2^{|Q|^2}$. Otherwise, all traces will be single states and multiplication will therefore commute on the trace semiring, but not on the relation semiring. So there cannot be an isomorphism. □

A homomorphism that sends path semirings to relation semirings can be built in the same way as ϕ_R, but using paths instead of a traces as an input. The homomorphism $\chi : 2^{A^*} \to 2^{A^*\times A^*}$ that sends language semirings to relation semirings uses a standard construction, the so-called *Cayley construction* (cf. [19]). It is defined, for all $L \subseteq A^*$ by

$$\tilde{\chi}(L) = \{(v, v.w) : v \in A^* \text{ and } w \in L\}.$$

Lemma 33. *Every identity that holds in all path or language semirings holds in all relation semirings.*

It is important to distinguish between relation semirings and relational structures under addition and multiplication.

Example 34. The relational structure from Example 15 is *not* a relation semiring: Its greatest element, which is the set $S = \{(n, n+k) : n, k \in \mathbb{N}\}$, differs from the greatest element $\mathbb{N} \times \mathbb{N}$ of any relation semiring over $\mathbb{N}$. Therefore, by definition, the example semiring is not a subalgebra of any relation semiring, thus not a relation semiring.

This fact explains the deviant behaviour of this model in Example 15 and in later sections.

13. Omega on Traces, Paths and Languages

In the previous sections we discussed star and omega on finite structures and presented two relational examples. We now study these operations on infinite structures like trace, path and language semirings. A main result is that omega can often be reduced to the star. In Section 15, we contrast this with the behaviour of divergence. We also study both operations on relation semirings in that section.

By definition, trace, path and language semirings are complete and satisfy all necessary distributivity laws. They are $*$-continuous; hence the star exists, can be computed iteratively, and coincides with the reflexive transitive closure. We therefore focus on the omega operation. But before proving an abstract reduction result, we consider an example.

Example 35. A set of traces T over P and A on a trace semiring can always be split in its test part $T_t = T \cap P$ and its testfree part $T_a = T - P$ such that $T = T_t + T_a$. This allows us to compute T_a^ω separately:

Since T_a is test-free, every trace $\tau \in T_a\top$ satisfies $|\tau| > 1$. Therefore, by induction, $|\tau| > n$ for all $\tau \in T_a^n\top$ and consequently $T_a^\omega \leq \inf(T_a^n\top : n \in \mathbb{N}) = \emptyset$.

The action part T_a^ω of T^ω therefore vanishes on all trace semirings, hence also on path and language semirings.

We will now transform this observation into an abstract result that relates the omega and the star on a class of i-semirings. It provides a necessary and sufficient condition to annihilate the omega of the testfree part of an action.

First, the following general fact can be proved by automated deduction. It allows us to replace $x^\omega = 0$ by an equivalent condition that does not mention the omega and therefore might be easier to check.

Lemma 36. *Let x, y be elements of some omega algebra. Then*

$$x^\omega = 0 \;\Leftrightarrow\; \forall y.(y \leq xy \Rightarrow y = 0).$$

Now suppose that — as in Example 35 — x can be split into its (maximal) *test part* x_t and its *testfree part* x_a:

$$x = x_t + x_a, \qquad x_t \in \mathsf{test}(S), \qquad (x_a)_t = (x_t)_a = 0.$$

Then Lemma 2(d) allows us to separate x_a^ω in the infinite iteration, since

$$x^\omega = (x_a + x_t)^\omega = x_a^\omega + x^* x_t \top.$$

By Lemma 36 we then know that x_a^ω vanishes iff all y that satisfy $y \leq x_a y$ vanish.

The following proposition collects conditions under which the testfree parts of *all* actions vanish.

Proposition 37. *Let S be an omega algebra in which all elements can be separated into test parts and testfree parts. Let $\top_a$ be the greatest testfree element of S. Then the following statements are equivalent.*

(a) $\forall x \in S.x^\omega = x^* x_t \top$.

(b) $\forall x \in S.x_a^\omega = 0$.

(c) $\forall x, y \in S.(y \leq x_a y \Rightarrow y = 0)$.

(d) $\forall y \in S.(y \leq \top_a y \Rightarrow y = 0)$.

The proofs have be automated. Again, (c) and (d) do not mention the omega; by (d) it even suffices to inspect $\top_a$. Intuitively, (d) says that 0 is the only element y which is below $\top_a y$, that is, for which multiplication with the most general action part $\top_a$ does not yield progress or length-increase. This condition abstractly captures the length-increase argument in Example 35. It is necessary and sufficient for annihilating the testfree parts of all omegas.

The splitting of arbitrary elements into their test and testfree part works for all i-semirings defined over Boolean algebras, such as trace, path, language and relation semirings. In these structures, $x_t = x \cap 1$, $x_a = x - x_t$ and $\top_a$ is the complement of 1 on the entire algebra.

Proposition 38. *In trace, path and language semirings, $x^\omega = x^* x_t \top$ for any element x. In language semirings, in particular, $x^\omega = A^*$ if $\varepsilon \in x$ and $\emptyset$ otherwise.*

Proof. We first consider trace semirings. As observed in Example 35, sets of traces T over P and A can always be partitioned into $T_t = T \cap P$, $T_a = T - P$ and $\top_a = (P, A)^* - P$. Since $\top_a$ is test-free, every trace $\tau \in \top_a$ satisfies $|\tau| > 1$. Therefore the multiplication $\top_a T$ increases the length of all traces and T cannot be contained in $\top_a T$ unless T is empty. Hence the equation $\forall T.(T \subseteq \top_a T \Rightarrow T = \emptyset)$ holds and the claim follows by Proposition 37.

For path and language semirings, the claim follow by the results of the previous sections. □

Interestingly, the proof uses neither completeness, infinite distributivity nor induction. The only condition needed is that the element $\top_a$ represents a kind of "proper progress" which cannot be undone.

Proposition 38 shows that in trace, path and language semirings omega can be explicitly defined by the star. Omega, which seemingly models infinite iteration, reduces to finite iteration after which a miracle (*anything*) happens. Moreover, if an element is purely testfree, its "infinite iteration" yields zero.

In relation semirings the situation is different: there is no notion of length that would increase through relative products or iterations and the condition $y \leq \top_a y \Rightarrow y = 0$ does not hold. For example, $\top_a \leq \top_a \top_a$ if the carrier set contains at least three elements. We therefore determine omega in relation semirings relative to a notion of divergence.

14. Divergence Semirings

Divergence can be axiomatised algebraically on i-semirings with additional modal operators. The definition of modal operators on semirings itself can be based on the axiomatisation of an antidomain operation, which is the Boolean complement of a domain operation.

A *(Boolean) domain semiring* [6] is a semiring S endowed with an *antidomain operation* $a : S \to S$ that satisfies

$$a(x)x = 0, \qquad a(xy) + a(xa^2(y)) = a(xa^2(y)), \qquad a^2(x) + a(x) = 1.$$

Every domain semiring is automatically idempotent. The image $a(S)$ of S under the antidomain function a forms a Boolean algebra with least element 0 and greatest element 1. This Boolean algebra is the maximal Boolean subalgebra in S that is bounded by 0 and 1. A *domain operation* $d : S \to S$ can be defined on S as $d(x) = a^2(x)$, and $d(x)$ and $a(x)$ are Boolean complements. $d(S) = a(S)$ therefore yields a suitable test algebra on S. It can also be shown that the elements of this test algebras are precisely the fixed points of the domain function: they satisfy $x = d(x)$. Finally, d and a correspond to the standard operations of domain and antidomain on trace, path, language and relation semirings. On trace semirings, for instance, the domain operation yields the set of all states which are starting points of traces, antidomain yields the complement of that set.

Diamond operators can now be defined for all $x \in S$ and $p \in d(S) = \mathsf{test}(S)$ as abstract preimages of p under x,

$$\langle x \rangle p = d(xp).$$

Intuitively, $\langle x \rangle p$ characterises the set of states with at least one x-successor in p. The modal operator $\langle x \rangle$ distributes over arbitrary suprema in $\mathsf{test}(S)$. The axiomatisation of modal semirings extends to modal Kleene algebras and modal omega algebras without any further axioms.

Codomain and anticodomain operations can be axiomatised dually with respect to semiring opposition. Dual modal operators can then be defined. In this setting, diamonds are lower adjoints of a Galois connection.

We will use the following properties of diamonds and domain: $\langle p \rangle q = pq$, $d(x) = 0 \Leftrightarrow x = 0$, $d(\top) = 1$, $d(p) = p$. Domain is isotone and diamonds are isotone in both arguments. A collection of automatically verified properties of domain and modal operators can be found at our web site [9].

A domain semiring S is a *divergence semiring* [5] if it can be endowed with a total operation $\nabla : S \to \mathsf{test}(S)$ that satisfies the ∇-unfold and ∇-coinduction axioms

$$\nabla x \leq \langle x \rangle \nabla x \quad \text{and} \quad p \leq \langle x \rangle p \Rightarrow p \leq \nabla x.$$

We call ∇x the *divergence* of x. This axiomatisation can be motivated on trace semirings as follows: The test $p - \langle x \rangle p$ characterises the set of x-maximal elements in p, that is, the set of elements in p from which no further x-action is possible. ∇x therefore has no x-maximal elements by the ∇-unfold axiom. By the ∇-coinduction axiom it is the greatest set with that property. It is easy to see that $\nabla x = 0$ iff x is Noetherian in the usual set-theoretic sense.

Divergence therefore captures the standard notion of program termination. All those states that admit only finite traces are characterised by the complement of ∇x. The operation of divergence can even be equationally axiomatised on domain semirings [22], but these axioms are of no relevance here.

Since diamonds are lower adjoints, fixed point fusion (Theorem 10) implies that the ∇-coinduction axiom is equivalent to

$$p \le q + \langle x \rangle p \Rightarrow p \le \nabla x + \langle x^* \rangle q,$$

which has the same structure as the omega coinduction axiom [5]. In particular, ∇x is the greatest fixed point of the function $\lambda y.\langle x \rangle y$, which corresponds to x^ω and $\nabla x + \langle x^* \rangle q$ is the greatest fixed point of the function $\lambda y.q + \langle x \rangle y$, which corresponds to $x^\omega + x^* y$. Moreover, the least fixed point of $\lambda y.q + \langle x \rangle y$ is $\langle x^* \rangle q$, which corresponds to $x^* y$. These fixed points are now defined on test algebras, which are Boolean algebras. A similar result for omega does not hold. Divergence therefore displays a more natural relationship between corecursion and its tail variant.

But again, iterative solutions of ∇x only exist when the test algebra is finite and all diamonds are defined. In general, modal diamonds do not distribute over arbitrary meets and therefore we can only approximate

$$\nabla x \le \inf(\langle x^n \rangle 1 : n \in \mathbb{N}) = \inf(d(x^n) : n \in \mathbb{N}).$$

However, the algebra A_3^2 shows that even finite i-semirings, which have a complete test algebra, need not be modal semirings (cf. Example 41 below).

We further need the following properties of divergence.

Lemma 39. *In every divergence semiring ∇ is isotone and*

$$\langle x \rangle \nabla x \le \nabla x, \qquad \nabla p = p, \qquad \nabla x \le d(x).$$

Additional properties can be found at our web site [9].

15. Divergence Across Models

We now relate omega and divergence in all models discussed so far. Concretely, we validate the two identities $(\nabla x)\top = x^\omega$ and $\nabla x = d(x^\omega)$ that arose from our motivating example in Section 8. We say that omega is *tame* if every element x satisfies the first identity; it is called *benign* if every element x satisfies the second one. We are also interested in the *taming condition* $d(x)\top = x\top$. All abstract results of this and the next section have been automatically verified by Prover9 or Mace4.

First, we consider these properties on relation semirings which we could not treat as special cases of trace semirings in Section 13. It is well known from relation algebra that all relation semirings satisfy the taming condition. The following section shows through abstract calculations that omega and divergence are related in relation semirings as expected and, as a special case, $x^\omega = 0$ iff x is Noetherian in relation semirings.

We now revisit the finite i-semirings of Examples 12 and 13.

Example 40. In the Boolean semiring, $d(0) = 0$ and $d(1) = 1$. Therefore, by Lemma 39, $\nabla 0 = 0$ and $\nabla 1 = 1$.

Example 41. In A_3^1 and A_3^3, the test algebra is always $\{0, 1\}$; $d(0) = 0$ and $d(1) = 1$. Moreover, by Lemma 39, $\nabla 0 = 0$ and $\nabla 1 = 1$. Setting $d(a) = 1 = \nabla a$ turns both into divergence semirings. In contrast, domain cannot be defined on A_3^2.

Consequently, omega is not tame in A_3^2, since $\nabla a \top$ is undefined in this model, and in A_3^3. However, it is tame in A_3^1 and A_2. In all four finite i-semirings, omega is benign.

Let us now consider trace, path and language semirings. Domain, diamond and divergence can indeed be defined on all these models. On a trace semiring,

$$d(T) = \{p : p \in P \text{ and } p.\sigma \in T \text{ for some } \sigma\}.$$

So, as expected, $\nabla T = \inf(d(T^n) : n \in \mathbb{N})$ characterises all states where infinite paths may start whenever P is finite. But since the omega operator is related to finite behaviours in all these models (cf. Lemma 38), the expected relationships to divergence fail.

Lemma 42. *The taming condition does not hold on some trace and path semirings. Omega is neither tame nor benign.*

Proof. Consider a trace semirings with $P = \{p\}$ and $A = \{a\}$ and let T be the set consisting of the single trace pap. Then $d(T) = \{p\} = \nabla T$ and $d(T)\top = \{p\}\top = \nabla(T)\top$ is the set of all non-empty traces over p and a. Moreover, $T\top = \{p.a.\tau : \tau \in (P, A)^*\}$. Finally, by Lemma 38($a$) $T^\omega = T^* T_t \top = \emptyset$ since $T_t = \emptyset$ in the example. This refutes all three identities for trace semirings. The argument translates to path semirings by forgetting actions. □

The situation for language semirings, where states are forgotten and the product operation is total, is different.

Lemma 43.

(a) *The taming condition does not hold in some language semirings.*

(b) *Omega is tame in all language semirings.*

(c) $(\nabla x)\top = x^\omega \not\Rightarrow d(x)\top = x\top$ *in some language semirings.*

Proof. In language semirings the test algebra is $\{\emptyset, \{\varepsilon\}\}$. So $d(L) = \{\varepsilon\}$ iff $L \neq 0$ for every $L \in 2^{A^*}$.

(a) Consider the language semiring over the single letter a and the language $L = \{a\}$. Then $d(L) = \{\varepsilon\}$ and therefore $d(L)\top = \top \neq L\top$, since $\varepsilon \in \top$, but $\varepsilon \notin L\top$.

(b) $\nabla L = \{\varepsilon\}$ iff $L \neq \emptyset$. Therefore $(\nabla L)\top = \top$ iff $L \neq \emptyset$ and $(\nabla L)\top = \emptyset$ iff $L = \emptyset$. It has already been shown in Lemma 38(b) that L^ω satisfies the same conditions.

(c) Immediate from (a) and (b). □

The next section shows that omega is benign on language semirings (without satisfying the taming condition).

Example 44. We now determine ∇x for $x = \{(n, n+1) : n \in \mathbb{N}\}$ from Example 15. Here, $\nabla x = \{(n, n) : n \in \mathbb{N}\}$ satisfies the divergence unfold and coinduction axioms. It has already been shown that $x^\omega = \emptyset$ (cf. Example 15). It immediately follows that omega is neither tame nor benign in this structure. It also does not satisfy the taming condition, since $d(x)\top = \top \neq \{(n, n+k) : k \geq 1\} = x\top$.

Remember that this relational structure is *not* a relation semiring, as pointed out in Example 34. This result therefore does not contradict any statement about relation semirings.

Therefore, omega is well behaved on relation semirings, but not on trace, language and path semirings. While relations are standard for finite input/output behaviour, traces, languages and paths are models for reactive and hybrid systems. By Proposition 38, omega can be expressed in these models by the finite iteration operator, hence does not precisely model proper infinite iteration. In contrast, the divergence operator models infinite behaviour in a natural and perhaps more appropriate way.

16. Taming the Omega

Our previous results certainly deserve a model-independent analysis. We henceforth briefly call *omega divergence semirings* a divergence semiring that is also an omega algebra. We now consider tameness of omega for this class. It is easy to show that the simple identities

$$x\top \leq d(x)\top, \quad x^\omega \leq (\nabla x)\top, \quad d(x^\omega) \leq \nabla x$$

hold in all omega divergence semirings. Therefore we only need to consider the relationships between their converses.

Proposition 45. *In the class of omega divergence semirings, the following implications hold, but not their converses.*

$$d(x)\top \leq x\top \;\Rightarrow\; (\nabla x)\top \leq x^\omega \;\Rightarrow\; \nabla x \leq d(x^\omega).$$

Proof. All implications are easy exercises in automated deduction. The converse of the first implication fails in the class of language semirings by Lemma 43(c). The converse of the second one fails in A_3^3 since $\nabla a = 1 = d(a) = d(a^\omega)$, but $(\nabla a)\top = 1 > a = a^\omega$ by Example 13 and 41. □

Proposition 45 shows that the tameness condition implies that omega is tame, which again implies that omega is benign. To complete the picture, we now consider the additional condition

$$d(x^\omega)\top = x^\omega\top = x^\omega$$

which is similar to the taming condition. Again, it is easy to show that $x^\omega \leq d(x^\omega)\top$.

Lemma 46. *In the class of omega divergence semirings,*

(a) $(\nabla x)\top = x^\omega \Rightarrow d(x^\omega)\top \leq x^\omega$,

(b) $\nabla x \leq d(x^\omega) \not\Rightarrow d(x^\omega)\top \leq x^\omega$,

(c) $\nabla x \leq d(x^\omega) \not\Leftarrow d(x^\omega)\top \leq x^\omega$.

Proof. (a) has been shown by automated deduction.

(b) In A_3^3, $\nabla a = 1 = d(a) = d(a^\omega)$ by Example 13 and 41. However, $d(a^\omega)\top = d(a)\top = 1 > a = a^\omega$. So it follows that $\nabla a \leq d(a^\omega) \not\Rightarrow d(a^\omega)\top \leq a^\omega$.

(c) By Example 44, $\nabla x \neq 0 = x^\omega = x^\omega\top$. It has already been shown that the underlying structure is a omega divergence semiring [5]. □

The remaining relationships between these conditions follow by transitivity. All relationships are shown in Figure 2.

Corollary 47. *In the class of omega divergence semirings,*

(a) $d(x)\top = x\top \Rightarrow \nabla x = d(x^\omega)$,

(b) $d(x)\top = x\top \not\Leftarrow \nabla x = d(x^\omega)$,

(c) $d(x)\top = x\top \Rightarrow d(x^\omega)\top = x^\omega$,

(d) $d(x)\top = x\top \not\Leftarrow d(x^\omega)\top = x^\omega$,

(e) $d(x^\omega)\top = x^\omega \not\Rightarrow (\nabla x)\top = x^\omega$.

This concludes our investigation of divergence and omega. These two notions of nontermination are unrelated in general. Properties that seem intuitive for relations can be refuted on three-element or natural infinite models. On relation semirings, omega seems consistent with the demonic view on total program correctness. On traces, paths and languages, it vanishes for pure actions that do not contain a test part. The taming condition that seems to play a crucial role could only be verified on (finite and infinite) relation semirings. All possible behaviours arise already for small finite models. Divergence is solidly founded on set-theoretic intuition. It behaves as expected on all models considered and therefore seems very promising for modelling infinite behaviours of programs and discrete dynamical systems.

17. Conclusion

We compared two notions of nontermination in the context of idempotent semirings: infinite iteration as modelled by the omega operator and divergence as defined on modal semirings. It turned out that divergence models the expected behaviour on standard computational models such as relations, traces, paths and languages. The omega operator, however, shows some anomalies. First, omega is not benign (whence not tame) on traces and paths, which are among the standard models for systems with infinite behaviours such as reactive and hybrid systems. Second, on traces, paths and languages, the omega reduces to the star, hence does not model infinite behaviour in a natural way. Third, whereas

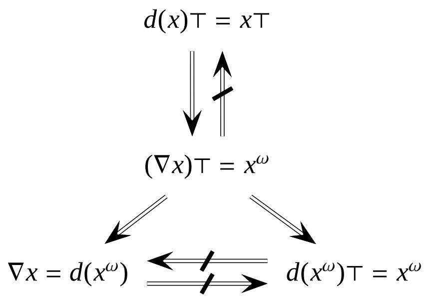

Figure 2: Relationships between a^ω and ∇a.

the axioms of divergence semirings can be explained in terms of tail corecursion and fixed point fusion, this is not possible for the omega, hence the omega algebra axioms do, to a certain extent, lack a broad intuitive motivation.

Our approach considers *infinite* behaviours on *finite* traces, words and paths. Nevertheless, divergence detects the correct infinite behaviour that arises from unravelling labelled transition systems, and it captures also models based on infinite traces, paths and words [22]. But omega algebras are by definition not appropriate for such infinite objects: The right zero axiom $x0 = 0$ excludes that x is an infinite element. It seems in general unreasonable to sequentially compose an infinite element x with another element y to xy. Two alternatives to omega algebras allow adding infinite elements: The weak omega algebras introduced by von Wright [23], and in particular the module-based structures introduced by Ésik and Kuich [7], in which finite and infinite elements have different sorts. But von Wright's axioms rule out the relational model. Omega algebras have recently been relativised to the module-based setting [22] and shown to correctly model the desired behaviours also on truly infinite models.

In conclusion, we believe that divergence semirings (and omega modules) have some potential for modelling and verifying finite and infinite behaviours of discrete dynamical systems. Studying notions such as stability, steady state, attractor in this modal setting and developing modal tools for analysing them could be a main direction of future research. Further potential applications include the development and verification of tools for the termination and nontermination analysis of programs, and the verification and refinement of concurrent, reactive and hybrid systems.

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A. Proof Template for Prover9/Mace4

The following template allows readers to repeat the calculational proofs in this paper with the theorem prover Prover9 and the search for counterexamples with the tool Mace4. The sos-part contains the axioms under consideration and, possibly, some additional assumptions. The goal-part contains the goal to be proved. Prover9 is refutationally complete for first-order logic with equality, that is, whenever the goal is a theorem in this logic, it will negate the goal, add it to the assumptions and derive the empty clause. If the goal is not a theorem, the prover might run forever. Complex theorems may require adding some appropriate lemmas. The theorems of this paper can usually be proved from the axioms alone in a couple of minutes, many of them almost instantaneously.

```
op(500, infix,   "+").   %addition
op(490, infix,   ";").   %multiplication
op(480, postfix, "*").   %star
op(470, postfix, "^").   %omega

formulas(sos).
   %  Idempotent semirings
        x+y=y+x  &  x+0=x  &  x+(y+z)=(x+y)+z.  %additive commutative monoid
        x;(y;z)=(x;y);z  &  x;1=x  &  1;x=x.    %multiplicative monoid
        0;x=0  &  x;0=0.                        %zero of multiplication
        x;(y+z)=x;y+x;z  &  (x+y);z=x;z+y;z.    %distributive laws
        x+x=x.                                  %idempotence
        x<=y <-> x+y=y.                         %definition of order
   %  Kleene algebras
        1+x;x*=x*  &  1+x*;x=x*.
        z+x;y<=y -> x*;z<=y  &  z+y;x<=y -> z;x*<=y.
   %  Antidomain
        a(x);x=0  &  a(x;y)=a(x;a(a(y)))  &  a(a(x))+a(x)=1.
   %  Domain
        d(x)=a(a(x)).
   %  Divergence
        d(x;div(x))=div(x)  &  d(y)<=d(x;d(y))+d(z) -> d(y)<=div(x)+d(x*;z).
   %  Omega
        x;x^=x^  &  z<=x;z+y -> z<=x^+x*;y.
   %  Additional laws
        test(x) <-> x=d(x).
        T=1^.
        x<=y -> d(x)<=d(y).
 end_of_list.

formulas(goals).
    % add goal here.
end_of_list.
```