Quantales and Temporal Logics

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Abstract We provide an algebraic semantics for the temporal logic CTL^* and simplify it for its sublogics CTL and LTL. We abstractly represent state and path formulas over transition systems in Boolean left quantales. These are complete lattices with an operation of multiplication that is completely disjunctive in its left argument and isotone in its right argument. On these quantales, the semantics of CTL^* formulas can be encoded via finite and infinite iteration operators, the CTL and LTL operators can be related to domain operators. This yields interesting new connections between representations as known from the modal μ-calculus and Kleene/ω-algebraic ones.

1 Introduction

The temporal logic CTL^* and its sublogics CTL and LTL are prominent tools in the analysis of parallel and reactive systems. Although they are by now well-understood, one rarely finds attempts to set up formal connections between them that go beyond mere inclusion of the sublogics into the overall logic. First results along these lines were obtained by B. von Karger in his work on temporal algebra [19]. But he stayed with implicit fixpoint characterisations of the involved semantic operators. In the present paper we show that we can map both CTL and LTL to closed expressions using modal operators as well as Kleene star and ω iteration.

This is achieved in two steps. First we provide an algebraic semantics for the full logic CTL^* on the basis of quantales, i.e., complete lattices with an operation of multiplication that is completely disjunctive in its left and positively disjunctive in its right argument.

In such a quantale, sets of states and hence the semantics of state formulas can be represented as test elements in the sense of Kozen [12], while general elements represent the semantics of path formulas.

We define suitable mappings that, for the CTL and LTL formulas, transform their general CTL^* semantics into simplified versions in ω-regular form. This yields interesting new connections between representations as known from the modal μ-calculus [10] and Kleene/ω-algebraic ones.

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2 Modelling CTL^*

In CTL^* (see e.g. [8]) one distinguishes path formulas and state formulas, the former ones denoting sets of computation traces and the latter ones denoting sets of states.

The language Ψ of *CTL* formulas* over a set Φ of atomic propositions is defined by the grammar

$$\Psi \ ::= \ \bot \mid \Phi \mid \Psi \to \Psi \mid \mathsf{X}\,\Psi \mid \Psi\,\mathsf{U}\,\Psi \mid \mathsf{E}\Psi,$$

where X and U are the next-time and until operators and E is the existential quantifier on paths. As usual,

$$\begin{array}{rlrl} \neg\varphi &=_{df}\ \varphi \to \bot, & \varphi \wedge \psi &=_{df}\ \neg(\varphi \to \neg\psi), \\ \varphi \vee \psi &=_{df}\ \neg\varphi \to \psi, & \mathsf{A}\varphi &=_{df}\ \neg\,\mathsf{E}\neg\varphi. \end{array}$$

The sublanguages Σ of *state formulas* and Π of *path formulas* are given by

$$\begin{array}{l} \Sigma ::= \bot \mid \Phi \mid \Sigma \to \Sigma \mid \mathsf{E}\Pi, \\ \Pi ::= \Sigma \mid \Pi \to \Pi \mid \mathsf{X}\,\Pi \mid \Pi\,\mathsf{U}\,\Pi. \end{array}$$

To motivate our algebraic semantics, we briefly recapitulate the standard semantics of CTL^* formulas. It uses a set S of states and traces $\sigma \in S^+ \cup S^\omega$ as its basic objects. By σ_i one denotes the i-th element of σ (numbering staring with 0) and by σ^i the trace that results from σ by removing its first i elements.

With each atomic proposition $\pi \in \Phi$ one associates the set $S_\pi \subseteq S$ of states for which p is true. Then one inductively defines when a formula φ *holds* for a trace σ, in signs $\sigma \models \varphi$:

$$\begin{array}{lll} \sigma \not\models \bot, & & \\ \sigma \models \pi & \text{iff} & \sigma_0 \in S_\pi, \\ \sigma \models \varphi \to \psi & \text{iff} & \sigma \models \varphi \text{ implies } \sigma \models \psi, \\ \sigma \models \mathsf{X}\,\varphi & \text{iff} & \sigma^1 \models \varphi, \\ \sigma \models \varphi\,\mathsf{U}\,\psi & \text{iff} & \exists\, j \geq 0\,.\, \sigma^j \models \psi \text{ and } \forall\, k < j\,.\, \sigma^k \models \varphi, \\ \sigma \models \mathsf{E}\varphi & \text{iff} & \exists\, \tau\,.\, \tau_0 = \sigma_0 \text{ and } \tau \models \varphi. \end{array}$$

This implies $\sigma \models \neg\varphi$ iff not $\sigma \models \varphi$.

From this semantics one can extract a set-based one by assigning to each formula φ the set $[\![\varphi]\!] =_{df} \{\sigma \mid \sigma \models \varphi\}$. This is the basis of the algebraic model to be given below.

We quickly repeat the proof of validity of the CTL^* axiom

$$\neg\mathsf{X}\varphi \leftrightarrow \mathsf{X}\neg\varphi, \tag{1}$$

since this will be crucial for the characterisation of the algebraic representation of X in Section 4:

$$\sigma \models \neg\mathsf{X}\varphi \ \Leftrightarrow\ \sigma \not\models \mathsf{X}\varphi \ \Leftrightarrow\ \sigma^1 \not\models \varphi \ \Leftrightarrow\ \sigma^1 \models \neg\varphi \ \Leftrightarrow\ \sigma \models \mathsf{X}\neg\varphi\ .$$

3 Quantales, Fixpoints and Modal Operators

Let us now transfer this to an algebraic setting. A *left quantale* [16] is a structure $(S, \leq, 0, \cdot, 1)$ where $(S, \leq)$ is a complete lattice and $\cdot$ is completely disjunctive in its left and positively disjunctive in its right argument. The infimum and supremum of two elements $a, b \in S$ are denoted by $a \sqcap b$ and $a + b$, resp. Both operators have equal binding power. The greatest element of S is denoted by $\top$. The definition implies that $\cdot$ is left-strict, i.e., that $0 \cdot a = 0$ for all $a \in S$.

A *right quantale* is defined symmetrically. Finally, $(S, \leq, 0, \cdot, 1)$ is a *quantale* if it is both a left and right one. In a (right) quantale multiplication is right-strict, i.e., $a \cdot 0 = 0$ for all $a \in S$. The notion of a quantale is equivalent that of a *standard Kleene algebra* [3].

A quantale is called *Boolean* if its underlying lattice is distributive and complemented, whence a Boolean algebra. An important Boolean quantale is REL, the algebra of binary relations under union and composition over a set.

We now introduce two important Boolean left quantales. Both are based on finite and infinite strings over an alphabet A. Next to their classical interpretation as characters, the elements of A may e.g. be thought of as states in a computation system, or, in connection with graph algorithms, as graph nodes. As usual, A^* is the set of all finite words over A; the empty word is denoted by ε. Moreover, A^ω is the set of all infinite words over A. We set $A^\infty =_{df} A^* \cup A^\omega$. The length of word s is $|s|$. As usual, concatenation is denoted by juxtaposition, where $st =_{df} s$ if $|s| = \infty$. A *language* over A is a subset of A^∞. As usual, we identify a singleton language with its only element. For language $S \subseteq A^\infty$ we define its infinite and finite parts by

$$\mathsf{inf}\, S =_{df} \{s \in S : |s| = \infty\}, \qquad \mathsf{fin}\, S =_{df} S - \mathsf{inf}\, S \ .$$

The left quantale $\mathrm{WOR}(A) = (\mathcal{P}(A^\infty), \subseteq, \emptyset, \ , \varepsilon)$ is obtained by extending concatenation to languages in the following way:

$$S\,T =_{df} \mathsf{inf}\, S \cup \{st : s \in \mathsf{fin}\, S \wedge t \in T\} \ .$$

Note that in general $S\,T \neq \{st : s \in S \wedge t \in T\}$; using the set on the right hand side as the definition of $S\,T$ one would obtain a right-strict operation. With the above definition, $S\,\emptyset = \mathsf{inf}\, S$ and hence $S\,\emptyset = \emptyset$ iff $\mathsf{inf}\, S = \emptyset$. It is straightforward to show that WOR(A) is an left quantale. The algebra is well-known from the classical theory of ω-languages (see e.g. [18] for a survey).

Besides this model we use a second one with a more refined view of composition and hence allows more interesting modal operators. It uses the *join* or *fusion product* $\bowtie$ of words as a language-valued operation. For $s \in A^*$, $t \in A^\infty$ and $x, y \in A$,

$$\begin{array}{lll} \varepsilon \bowtie \varepsilon =_{df} \varepsilon & \varepsilon \bowtie s =_{df} s \bowtie \varepsilon =_{df} \emptyset & \text{if } s \neq \varepsilon\ , \\ sx \bowtie xt =_{df} sxt\ , & sx \bowtie yt =_{df} \emptyset & \text{if } x \neq y\ . \end{array}$$

Finally, $s \bowtie t =_{df} s$ if $|s| = \infty$.

Informally, a non-empty finite word s can be joined with a non-empty word t iff the last letter of s coincides with the first one of t; only one copy of that letter is kept in the joined word.

Since we view the infinite words as streams of computations, we call the left quantale based on this composition operation $\mathrm{STR}(A)$ and define it by $\mathrm{STR}(A) =_{df} (\mathcal{P}(A^\infty), \subseteq, \emptyset, \bowtie, A \cup \varepsilon)$, where $\bowtie$ is extended to languages in the following way:

$$S \bowtie T =_{df} \mathsf{inf}\, S \cup \{s \bowtie t : s \in \mathsf{fin}\, S \wedge t \in T\} \ .$$

As above, we have $S \bowtie \emptyset = \mathsf{inf}\, S$ and hence $S \bowtie \emptyset = \emptyset$ iff $\mathsf{inf}\, S = \emptyset$. A transition relation can be modelled in STR as a set R of words of length 2. The powers R^i of R then consist of the words (or paths) of length $i + 1$ that are generated by R-transitions.

Arbitrary finite and infinite iteration are defined in a quantale by the usual recursions:

$$a^* =_{df} \mu x \,.\, 1 + a \cdot x \ , \qquad a^\omega =_{df} \nu x \,.\, a \cdot x \ .$$

If, as in a Boolean quantale, $+$ is completely conjunctive then, as shown in [1], these operations satisfy the axioms of a left Kleene/omega algebra [11,2]. The two operations are connected as follows (see e.g. [1]):

$$a^* \cdot b = \mu x \,.\, b + a \cdot x \ , \qquad a^\omega + a^* \cdot b = \nu x \,.\, b + a \cdot x \ . \tag{2}$$

To model state formulas we use the idea of tests as introduced into Kleene algebras by Kozen [12]. Based on the observation that, relationally, a set of elements can be modelled as a subset of the identity relation, one defines a *(left) test quantale* as a pair $(S, \mathsf{test}(S))$, where S is a (left) quantale and $\mathsf{test}(S) \subseteq [0, 1]$ is a Boolean subalgebra of the interval $[0, 1]$ of S such that $0, 1 \in \mathsf{test}(S)$ and join and meet in $\mathsf{test}(S)$ coincide with $+$ and $\cdot$. We use $a, b, \dots$ for general quantale elements and $p, q, \dots$ for tests. By $\neg p$ we denote the complement of p in $\mathsf{test}(S)$ and set $p \to q = \neg p + q$. We freely use the Boolean laws for tests.

A set of states will now abstractly be represented by a test. Pre- and post-multiplication by a test correspond to restricting an element on the input and output side, resp. This allows us to represent the set of all possible paths that start with a state in set p by the *test ideal* $p \cdot \top$.

The set of starting states of paths in a set represented by $a \in S$ can be retrieved by the *domain operation* $\ulcorner\_ : S \to \mathsf{test}(S)$ characterised by the Galois connection

$$\ulcorner a \le p \Leftrightarrow a \le p \cdot \top \ .$$

This is well defined, since in a Boolean left quantale $\cdot$ preserves arbitrary infima of tests in its left argument [4]. By the general properties of Galois connections, the domain operation is completely disjunctive. For further domain properties see [5].

We list a number of important properties of tests and test ideals; for the proofs see [13].

Lemma 3.1 *Assume a Boolean left test quantale S and consider elements $a, b \in S$ and $p, q \in \mathsf{test}(S)$.*

1. $\ulcorner(p \cdot \top) = p$.
2. $p \leq q \Leftrightarrow p \cdot \top \leq q \cdot \top$.
3. $p \cdot a \sqcap b = p \cdot (a \sqcap b)$.
   Hence also $p \cdot \top \sqcap a = p \cdot a$ *and* $p \cdot (a \sqcap b) = p \cdot a \sqcap p \cdot b$.
4. $p \cdot a \sqcap q \cdot a = p \cdot q \cdot a$.
5. *If S is Boolean then* $\neg p \cdot \top = \overline{p \cdot \top}$.

By property 2. the set of test ideals is isomorphic to the set of tests.

To use the above properties freely, we assume for the remainder that S is a Boolean left quantale.

Using domain, we can also define (forward) modal operators by setting, for $a \in S$ and $q \in \mathsf{test}(S)$,

$$\langle a \rangle q =_{df} \ulcorner(a \cdot q) \ , \qquad [a] =_{df} \neg\langle a \rangle \neg q \ .$$

The box generalises the notion of the weakest liberal precondition wlp to Boolean left quantales. If we view a as the transition relation of a command then the test $[a]p$ characterises those states from which no transition under a is possible or the execution of a is guaranteed to end up in a final state that satisfies test q. Both operators are isotone in their test argument. Hence in a Boolean quantale we have the full power of the modal μ-calculus [10] available.

In particular, we can define the *convergence* $\triangle a \in \mathsf{test}(S)$ of an element a by

$$\triangle a =_{df} \mu x \,.\, [a]x \ .$$

This characterises the set of states from which no infinite transition paths emerge.

To make the modal operators well-behaved w.r.t. composition we need to assume the underlying quantale to satisfy

$$\ulcorner(a \cdot b) = \ulcorner(a \cdot \ulcorner b), \tag{3}$$

since then $\langle a \cdot b \rangle = \langle a \rangle \circ \langle b \rangle$ and $[a \cdot b] = [a] \circ [b]$. Therefore we call a (left) quantale with this property *modal*.

In a modal left quantale, star, omega, box and convergence interact according to the following induction and coinduction laws [5,6]:

$$x \leq p \cdot [a]x \Rightarrow x \leq [a^*]p, \tag{4}$$

$$\triangle a \cdot [a^*]p = \mu x \,.\, p \cdot [a]x. \tag{5}$$

Dual laws hold for the diamond operator.

In modal quantales (and, more generally, modal ω/convergence algebras) we have additional flexibility compared to PDL [10] and the μ-calculus, since the modal operators are defined for ω-regular expressions, not only for atomic actions.

4 Algebraic Semantics of CTL^*

Now we give our algebraic interpretation of CTL^* over a Boolean modal quantale S. To save some notation we set $\Phi = \mathsf{test}(S)$. Moreover, we fix an element a as representing the transition system underlying the logic. The precise requirements for a will be discussed in Section 5. Then the concrete semantics above generalises to a function $[\![\_]\!] : \Psi \to S$:

$$\begin{aligned}
[\![\bot]\!] &= 0, \\
[\![p]\!] &= p \cdot \top, \\
[\![\varphi \to \psi]\!] &= \overline{[\![\varphi]\!]} + [\![\psi]\!], \\
[\![\mathsf{X}\,\varphi]\!] &= a \cdot [\![\varphi]\!], \\
[\![\varphi\,\mathsf{U}\,\psi]\!] &= \bigsqcup_{j\geq 0} (a^j \cdot [\![\psi]\!] \sqcap \bigsqcap_{k<j} a^k \cdot [\![\varphi]\!]), \\
[\![\mathsf{E}\varphi]\!] &= \ulcorner [\![\varphi]\!] \cdot \top .
\end{aligned}$$

Using these definitions, it is straightforward to check that

$$[\![\varphi \vee \psi]\!] \;=\; [\![\varphi]\!] + [\![\psi]\!], \qquad [\![\varphi \wedge \psi]\!] \;=\; [\![\varphi]\!] \sqcap [\![\psi]\!], \qquad [\![\neg\varphi]\!] \;=\; \overline{[\![\varphi]\!]}.$$

An important check of the adequacy of our definitions is provided by the following theorem.

Theorem 4.1 *The element $[\![\varphi\,\mathsf{U}\,\psi]\!]$ is the least fixpoint μf of the function $f(x) =_{df} [\![\psi]\!] + ([\![\varphi]\!] \sqcap a \cdot x)$.*

Proof. Since in a Boolean quantale multiplication and meet are completely disjunctive, f is completely disjunctive, too, and hence continuous. So by Kleene's fixpoint theorem $\mu f = \bigsqcup_{j\geq 0} f^j(0)$. A straightforward induction that goes through provided multiplication by a distributes from the left through meet shows

$$f^i(0) = \bigsqcup_{j\leq i} (a^j \cdot [\![\psi]\!] \sqcap \bigsqcap_{k<j} a^k \cdot [\![\varphi]\!]),$$

from which the claim follows. That a has the mentioned property will be discussed in the next section. □

We define the usual abbreviations:

$$\mathsf{A}\varphi \;=_{df}\; \neg\mathsf{E}\neg\varphi, \qquad \mathsf{F}\varphi \;=_{df}\; \top\mathsf{U}\varphi, \qquad \mathsf{G}\varphi \;=_{df}\; \neg\mathsf{F}\neg\varphi.$$

The above theorem and (2) yield the following closed representation of F:

Corollary 4.2 $[\![\mathsf{F}\varphi]\!] \;=\; a^* \cdot [\![\varphi]\!]$.

5 The Next-Time Operator

We now want to find out the suitable requirements on a by transferring the axiom (1) to the algebraic frame. To satisfy it, we need to have for all formulas φ and their semantical values $b =_{df} [\![\varphi]\!]$,

$$\overline{a \cdot b} = [\![\neg \mathsf{X}\varphi]\!] = [\![\mathsf{X}\neg\varphi]\!] = a \cdot \overline{b}. \tag{6}$$

This semantic property can equivalently be characterised as follows (property 1. was already shown in [4]).

Lemma 5.1 *Consider a Boolean left quantale S and $a \in S$ such that $a \cdot 0 = 0$.*

1. $\forall\, b \in S : a \cdot \overline{b} \leq \overline{a \cdot b} \Leftrightarrow \forall\, b, c \in S : a \cdot (b \sqcap c) = a \cdot b \sqcap a \cdot c.$
2. $\forall\, b \in S : \overline{a \cdot b} \leq a \cdot \overline{b} \Leftrightarrow a \cdot \top = \top \Leftrightarrow a^\omega = \top.$

Proof. 1. ($\Rightarrow$) It suffices to show ($\geq$), since the reverse inequality follows by isotony.

$$\begin{array}{ll} & a \cdot b \sqcap a \cdot c \leq a \cdot (b \sqcap c) \\ \Leftrightarrow & \quad \{\!\![\text{ by shunting }]\!\!\} \\ & a \cdot b \leq \overline{a \cdot c} + a \cdot (b \sqcap c) \\ \Leftarrow & \quad \{\!\![\text{ assumption } a \cdot \overline{b} \leq \overline{a \cdot b}]\!\!\} \\ & a \cdot b \leq a \cdot \overline{c} + a \cdot (b \sqcap c) \\ \Leftrightarrow & \quad \{\!\![\text{ distributivity }]\!\!\} \\ & a \cdot b \leq a \cdot (\overline{c} + (b \sqcap c)) \\ \Leftrightarrow & \quad \{\!\![\text{ Boolean algebra }]\!\!\} \\ & a \cdot b \leq a \cdot (\overline{c} + b) \\ \Leftrightarrow & \quad \{\!\![\text{ lattice algebra and isotony }]\!\!\} \\ & \text{TRUE.} \end{array}$$

($\Leftarrow$) We calculate, using the assumption in the third step:

$$0 = a \cdot 0 = a \cdot (b \sqcap \overline{b}) = a \cdot b \sqcap a \cdot \overline{b}.$$

Now the claim is immediate by shunting.

2.
$$\begin{array}{ll} & \overline{a \cdot b} \leq a \cdot \overline{b} \\ \Leftrightarrow & \quad \{\!\![\text{ by shunting }]\!\!\} \\ & \top \leq a \cdot b + a \cdot \overline{b} \\ \Leftrightarrow & \quad \{\!\![\text{ distributivity }]\!\!\} \\ & \top \leq a \cdot (b + \overline{b}) \\ \Leftrightarrow & \quad \{\!\![\text{ complement }]\!\!\} \\ & \top \leq a \cdot \top \\ \Leftrightarrow & \quad \{\!\![\text{ greatest element }]\!\!\} \\ & \top = a \cdot \top \\ \Leftrightarrow & \quad \{\!\![a^\omega = \nu x \,.\, a \cdot x]\!\!\} \\ & a^\omega = \top. \end{array}$$

□

In relation algebra, the special case $a \cdot \overline{1} \le \overline{a}$ of the property in 1. characterises partial functions and is equivalent to the full property [17]. But in general quantales the special and the general case are not equivalent [4]. Moreover, again from [4], we know that in quantales such as LAN and PAT an element a is left-distributive over meet iff it is prefix-free, i.e. if no member of a is a prefix of another member. This holds in particular if all words in a have equal length, which is the case if a models a transition relation and hence consists only of words of length 2. The equivalent condition $\forall\, b\,.\, a \cdot b \sqcap a \cdot \overline{b} = 0$ was used in the computation calculus of R.M. Dijkstra [7].

But what about property 2? Only rarely will a quantale be "generated" by an element a in the sense that $a^\omega = \top$. The solution is to choose a left-distributive element a and restrict the set of semantical values to the subset $\mathrm{SEM}(a) =_{df} \{b : b \le a^\omega\}$, taking complements relative to a^ω. This set is clearly closed under $+$ and $\sqcap$ and under prefixing by a, since by isotony

$$a \cdot b \le a \cdot a^\omega = a^\omega \ .$$

Finally, it also contains all elements $p \cdot a^\omega$ with $p \in \mathsf{test}(S)$, since $p \le 1$. Hence the above semantics is well-defined in $\mathrm{SEM}(a)$ if we replace $\top$ by a^ω.

6 The Semantics of State Formulas

In this section we relate state formulas and test ideals.

Theorem 6.1 *Let φ be a state formula of CTL^*.*

1. *$[\![\varphi]\!]$ is a test ideal, and hence, by Lemma 3.1.1, we have $[\![\varphi]\!] = \ulcorner[\![\varphi]\!] \cdot \top$.*
2. *$[\![\mathsf{E}\varphi]\!] = [\![\varphi]\!]$.*
3. *$[\![\mathsf{A}\varphi]\!] = \neg\ulcorner(\overline{[\![\varphi]\!]}) \cdot \top$.*

Proof. 1. The proof is by induction on the structure of φ.

- For $\bot$ and $p \in \mathsf{test}(S)$ this is immediate from the definition.
- Assume that the claim already holds for state formulas φ and ψ. We calculate, using the definitions, the induction hypothesis, Lemma 3.1.5, distributivity and the definitions again,

$$[\![\varphi \to \psi]\!] = \overline{[\![\varphi]\!]} + [\![\psi]\!] = \overline{\ulcorner[\![\varphi]\!] \cdot \top} + \ulcorner[\![\psi]\!] \cdot \top = \neg\ulcorner[\![\varphi]\!] \cdot \top + \ulcorner[\![\psi]\!] \cdot \top$$
$$= (\neg\ulcorner[\![\varphi]\!] + \ulcorner[\![\psi]\!]) \cdot \top = (\ulcorner[\![\varphi]\!] \to \ulcorner[\![\psi]\!]) \cdot \top.$$

- For $\mathsf{E}\varphi$ the claim is immediate from the definition.

2. Immediate from 1. and the definition of $[\![\mathsf{E}\varphi]\!]$.
3. Similar. □

Moreover, state formulas are closed under $\neg, \wedge, \vee$ and A.

Next, we derive some properties of U and its relatives for state formulas. For this we need some knowledge about dual functions and their fixpoints. The *dual* f° of a function $f : S \to S$ over a Boolean quantale is, as usual, defined by $f^\circ(x) =_{df} \overline{f(\overline{x})}$. Then $\mu f = \overline{\nu f^\circ}$ and $\nu f = \overline{\mu f^\circ}$.

Lemma 6.2 *Let φ, ψ be state formulas and $p \cdot \top =_{df} [\![\varphi]\!], q \cdot \top =_{df} [\![\psi]\!]$.*

1. $[\![\varphi \mathsf{U} \psi]\!] \;=\; (p \cdot a)^* \cdot q \cdot \top \;=\; ([\![\varphi]\!] \sqcap a)^* \cdot [\![\psi]\!]$.

2. $[\![\mathsf{G}\varphi]\!] \;=\; (p \cdot a)^\omega \;=\; ([\![\varphi]\!] \sqcap a)^\omega$.
Hence we have the shunting rule $(p \cdot a)^\omega \;=\; \overline{a^* \cdot \neg p \cdot \top}$.

Proof. 1. Using Theorem 4.1 and Lemma 3.1.3 we calculate

$$[\![\varphi \mathsf{U} \psi]\!] \;=\; \mu x \,.\, q \cdot \top + (p \cdot \top \sqcap a \cdot x) \;=\; \mu x \,.\, q \cdot \top + p \cdot a \cdot x,$$

and the claim follows by (2).

2. Since $[\![\mathsf{F}\varphi]\!] \;=\; \mu f_p$ where $f_p(x) \;=\; p \cdot \top + a \cdot x$, we have, by Lemma 3.1, $[\![\mathsf{G}\varphi]\!] = [\![\neg\mathsf{F}\neg\varphi]\!] = \nu f^\circ_{\neg p}$, where, again by Lemma 3.1 and by (6),

$$f^\circ_{\neg p}(x) \;=\; \overline{\neg p \cdot \top + a \cdot \overline{x}} \;=\; \overline{\neg p \cdot \top} \sqcap \overline{a \cdot \overline{x}} \;=\; p \cdot \top \sqcap a \cdot x \;=\; p \cdot a \cdot x.$$

Hence the claim follows by the definition of ω. □

The case $p = 1$ yields again Corollary 4.2. Now we deal with E.

Lemma 6.3 $[\![\mathsf{EX}\varphi]\!] = [\![\mathsf{EXE}\varphi]\!]$.

Proof. Using the definitions, a domain property, (3) and the definitions again, we calculate

$$[\![\mathsf{EXE}\varphi]\!] \;=\; {}^\ulcorner(a \cdot {}^\ulcorner[\![\varphi]\!] \cdot \top) \cdot \top \;=\; {}^\ulcorner(a \cdot {}^\ulcorner[\![\varphi]\!]) \cdot \top \;=\; {}^\ulcorner(a \cdot [\![\varphi]\!]) \cdot \top \;=\; [\![\mathsf{EX}\varphi]\!].$$

□

Next, we collect a number of properties of A; the proofs are straightforward calculations.

Lemma 6.4

$$\begin{array}{rclr} [\![\mathsf{A}\bot]\!] &=& 0, & [\![\mathsf{A}\top]\!] = \top, \\ [\![\mathsf{A}(p \vee \varphi)]\!] &=& p + [\![\mathsf{A}\varphi]\!], & [\![\mathsf{A}(p \wedge \varphi)]\!] = p \cdot [\![\mathsf{A}\varphi]\!]. \end{array}$$

In particular, $[\![\mathsf{A}p]\!] = p$.

Moreover, for the axiom $\mathsf{EX}\top$ we obtain

Lemma 6.5 $[\![\mathsf{EX}\top]\!] = \top \;\Leftrightarrow\; {}^\ulcorner a = 1 \;\Leftrightarrow\;$ *a total.*

Proof. Since $[\![\mathsf{EX}\top]\!] = {}^\ulcorner(a \cdot \top) \cdot \top = {}^\ulcorner a \cdot \top$, the claim follows from Lemma 3.1.2. □

7 From CTL^* to CTL

For a number of applications the sublogic CTL of CTL^* suffices. We will see that it can be modelled in plain Kleene/convergence algebra. Syntactically, CTL consists of those CTL^* state formulas that result by using the restricted path formulas generated by the grammar $\Pi \ ::= \ \mathsf{X}\,\Sigma \mid \Sigma\,\mathsf{U}\,\Sigma$.

First, we note that EX and AX are duals.

Lemma 7.1 $\llbracket \mathsf{AX}\varphi \rrbracket \ = \ \llbracket \neg\mathsf{EX}\neg\varphi \rrbracket$.

Proof. By Theorem 6.1.3, the definitions, (6), Lemma 3.1.5 and the definitions again, we obtain

$$\begin{aligned}\llbracket \mathsf{AX}\varphi \rrbracket \ &= \ \neg\,\ulcorner\overline{\llbracket \mathsf{X}\varphi \rrbracket}\cdot\top \ = \ \neg\,\ulcorner\overline{a\cdot\llbracket\varphi\rrbracket}\cdot\top \ = \ \neg\,\ulcorner(a\cdot\overline{\llbracket\varphi\rrbracket})\cdot\top \\ &= \ \overline{\ulcorner(a\cdot\overline{\llbracket\varphi\rrbracket})\cdot\top} \ = \ \llbracket\neg\mathsf{EX}\neg\varphi\rrbracket.\end{aligned}$$

□

¿From this and Lemma 6.3 we obtain

Corollary 7.2 $\llbracket \mathsf{AX}\varphi \rrbracket = \llbracket \mathsf{AXA}\varphi \rrbracket$.

Since we already know that the semantics of every state formula φ is a test ideal, we can, by Theorem 6.1, use the simplified semantics $\llbracket\varphi\rrbracket_d$ given by

$$\llbracket\varphi\rrbracket_d \ =_{df} \ \ulcorner\llbracket\varphi\rrbracket\,.$$

This way we only need to calculate with tests.

By disjunctivity of Domain and Lemma 3.1,

$$\llbracket\varphi\vee\psi\rrbracket_d \ = \ \llbracket\varphi\rrbracket_d + \llbracket\psi\rrbracket_d, \qquad \llbracket\varphi\wedge\psi\rrbracket_d \ = \ \llbracket\varphi\rrbracket_d\cdot\llbracket\psi\rrbracket_d, \qquad \llbracket\neg\varphi\rrbracket_d \ = \ \neg\llbracket\varphi\rrbracket_d.$$

We transfer the properties of A from Lemma 6.4 to the simplified semantics; again the proofs are straightforward calculations.

Lemma 7.3

$$\begin{aligned}\llbracket\mathsf{A}\bot\rrbracket_d \ &= \ 0, & \llbracket\mathsf{A}\top\rrbracket_d &= 1, \\ \llbracket\mathsf{A}(p\vee\varphi)\rrbracket_d \ &= \ p\vee\llbracket\mathsf{A}\varphi\rrbracket_d, & \llbracket\mathsf{A}(p\wedge\varphi)\rrbracket_d &= p\cdot\llbracket\mathsf{A}\varphi\rrbracket_d.\end{aligned}$$

In particular, $\llbracket\mathsf{A}p\rrbracket_d = p$.

Now we can calculate the inductive behaviour of $\llbracket\_\rrbracket_d$.

Theorem 7.4

1. $\llbracket\bot\rrbracket_d = 0$,
2. $\llbracket p\rrbracket_d = p$,
3. $\llbracket\varphi\rightarrow\psi\rrbracket_d = \llbracket\varphi\rrbracket_d\rightarrow\llbracket\psi\rrbracket_d$,
4. $\llbracket\mathsf{EX}\varphi\rrbracket_d = \langle a\rangle\llbracket\varphi\rrbracket_d$,
5. $\llbracket\mathsf{AX}\varphi\rrbracket_d = [a]\llbracket\varphi\rrbracket_d \ = \ \llbracket\mathsf{AXA}\varphi\rrbracket_d$,
6. $\llbracket\mathsf{AF}\varphi\rrbracket_d = \neg\,\ulcorner\overline{a^*\cdot\llbracket\varphi\rrbracket_d\cdot\top} \ = \ \neg\,\ulcorner(\neg\llbracket\varphi\rrbracket_d\cdot a)^\omega$,
7. $\llbracket\mathsf{E}(\varphi\mathsf{U}\psi)\rrbracket_d = \langle(\llbracket\varphi\rrbracket_d\cdot a)^*\rangle\llbracket\psi\rrbracket_d$,
8. $\llbracket\mathsf{A}(\varphi\mathsf{U}\psi)\rrbracket_d = \llbracket A\mathsf{F}\varphi\rrbracket_d\cdot[b^*](\llbracket\varphi\rrbracket_d+\llbracket\psi\rrbracket_d)$ *where* $b =_{df} \neg\llbracket\varphi\rrbracket_d\cdot a$.

Proof. The proof is again by induction on the structure of the state formulas. The cases 1.–3. of $\bot, p$ and $\varphi \to \psi$ have already been covered in the proof of Theorem 6.1.

4. Using again Theorem 6.1, the definition of $[\![\,]\!]$, (3) and the definitions again, we calculate

$$[\![\mathsf{EX}\varphi]\!]_d \;=\; \ulcorner[\![\mathsf{X}\varphi]\!] \;=\; \ulcorner(a \cdot [\![\varphi]\!]) \;=\; \ulcorner(a \cdot \ulcorner[\![\varphi]\!]) \;=\; \langle a \rangle [\![\varphi]\!]_d .$$

5.
$$
\begin{array}{ll}
 & [\![\mathsf{AX}\varphi]\!]_d \\
= & \quad \{\![\text{ by Theorem 6.1.3 and Lemma 3.1.2 }]\!\} \\
 & \neg\ulcorner\overline{[\![\mathsf{X}\varphi]\!]} \\
= & \quad \{\![\text{ definition and Theorem 6.1 }]\!\} \\
 & \neg\ulcorner\overline{a \cdot [\![\varphi]\!]_d \cdot \top} \\
= & \quad \{\![\text{ by (6) }]\!\} \\
 & \neg\ulcorner(a \cdot \overline{[\![\varphi]\!]_d \cdot \top}) \\
= & \quad \{\![\text{ by Lemma 3.1.2 }]\!\} \\
 & \neg\ulcorner(a \cdot \neg[\![\varphi]\!]_d \cdot \top) \\
= & \quad \{\![\text{ domain property }]\!\} \\
 & \neg\ulcorner(a \cdot \neg[\![\varphi]\!]_d) \\
= & \quad \{\![\text{ definition }]\!\} \\
 & [a][\![\varphi]\!]_d .
\end{array}
$$

   Moreover, $[\![\varphi]\!]_d = [\![\mathsf{A}\varphi]\!]_d$ follows from Lemma 7.3.
6. Assume $[\![\varphi]\!] = p \cdot \top$. By the definition of A and the explicit representation of F from Corollary 4.2 we obtain $[\![\mathsf{AF}\varphi]\!] \;=\; \neg\ulcorner\overline{a^* \cdot p \cdot \top} \cdot \top$. Now the claim is immediate from the shunting rule of Lemma 6.2.2 and the definition of $[\![\,]\!]_d$.
7. For $[\![\mathsf{E}(\varphi \mathsf{U} \psi)]\!]$ we use the principle of *least-fixpoint fusion* [1]: If h is completely disjunctive and $h \circ f \;=\; g \circ h$ then $h(\mu f) \;=\; \mu g$.
   Set, for abbreviation, $p \;=_{df}\; [\![\varphi]\!]_d$ and $q \;=_{df}\; [\![\psi]\!]_d$. Then, by Lemma 4.1 and Lemma 3.1.3, $u \;=_{df}\; [\![\varphi \mathsf{U} \psi]\!] \;=\; \mu f$ where $f(x) \;=_{df}\; q \cdot \top + (p \cdot a \cdot x)$. Second, by Theorem 6.1 and (5), $\langle (p \cdot a)^* \rangle \;=\; \mu g$ where $g(p) \;=_{df}\; q + \langle (p \cdot a) \rangle p$. We need to show $\ulcorner(\mu f) \;=\; \mu g$. By the principle of least-fixpoint fusion this is implied by $\ulcorner \circ f \;=\; g \circ \ulcorner$, since $\ulcorner$ is completely disjunctive. We calculate:

$$
\begin{array}{ll}
 & \ulcorner(f(x)) \\
= & \quad \{\![\text{ definition } f \text{ }]\!\} \\
 & \ulcorner(q \cdot \top + (p \cdot a \cdot x)) \\
= & \quad \{\![\text{ additivity of domain }]\!\} \\
 & \ulcorner(q \cdot \top) + \ulcorner(p \cdot a \cdot x)) \\
= & \quad \{\![\text{ by Lemma 3.1.1 }]\!\} \\
 & q + \ulcorner(p \cdot a \cdot x) \\
= & \quad \{\![\text{ (3) }]\!\} \\
 & q + \ulcorner(p \cdot a \cdot \ulcorner x) \\
= & \quad \{\![\text{ definition diamond }]\!\}
\end{array}
$$

$$q + \langle p \cdot a \rangle \cdot {}^{\ulcorner}x$$

$= \quad \{\!\![$ definition g $]\!\!\}$

$$g({}^{\ulcorner}x).$$

8. For $r =_{df} [\![\mathsf{A}(\varphi \mathsf{U} \psi)]\!]$ we use that, by Theorem 6.1.3, $r = \neg {}^{\ulcorner}\overline{u}$, where $u =_{df} [\![\varphi \mathsf{U} \psi]\!]$. Let, for abbreviation, $p \cdot \top =_{df} [\![\varphi]\!]$ and $q \cdot \top =_{df} [\![\psi]\!]$. Since $u = \mu f$ where $f(x) = q \cdot \top + p \cdot a \cdot x$, we have $\overline{u} = \nu f^{\circ}$. We calculate

$$f^{\circ}(x)$$

$= \quad \{\!\![$ definitions $]\!\!\}$

$$\overline{q \cdot \top + p \cdot a \cdot \overline{x}}$$

$= \quad \{\!\![$ de Morgan $]\!\!\}$

$$\overline{q \cdot \top} \sqcap \overline{p \cdot a \cdot \overline{x}}$$

$= \quad \{\!\![$ by Lemma 3.1.5 $]\!\!\}$

$$\neg q \cdot \top \sqcap \overline{p \cdot \top \sqcap a \cdot \overline{x}}$$

$= \quad \{\!\![$ by Lemma 3.1.3 and de Morgan $]\!\!\}$

$$\neg q \cdot (\overline{p \cdot \top} + \overline{a \cdot \overline{x}})$$

$= \quad \{\!\![$ by Lemma 3.1.5 and (6) $]\!\!\}$

$$\neg q \cdot (\neg p \cdot \top + \overline{\overline{a \cdot x}})$$

$= \quad \{\!\![$ complement $]\!\!\}$

$$\neg q \cdot (\neg p \cdot \top + a \cdot x)$$

$= \quad \{\!\![$ distributivity $]\!\!\}$

$$\neg q \cdot \neg p \cdot \top + \neg q \cdot a \cdot x$$

$= \quad \{\!\![$ de Morgan $]\!\!\}$

$$\neg(p + q) \cdot \top + \neg q \cdot a \cdot x.$$

Hence

$$r$$

$= \quad \{\!\![$ above considerations $]\!\!\}$

$$\neg {}^{\ulcorner}(\nu f^{\circ})$$

$= \quad \{\!\![$ by (2) $]\!\!\}$

$$\neg {}^{\ulcorner}((\neg q \cdot a)^{\omega} + (\neg q \cdot a)^{*} \cdot \neg(p + q) \cdot \top)$$

$= \quad \{\!\![$ distributivity and de Morgan $]\!\!\}$

$$\neg {}^{\ulcorner}((\neg q \cdot a)^{\omega}) \cdot \neg {}^{\ulcorner}((\neg q \cdot a)^{*} \cdot \neg(p + q) \cdot \top)$$

$= \quad \{\!\![$ by Lemma 6.2.2 and domain property $]\!\!\}$

$$\neg {}^{\ulcorner}(\overline{a^{*} \cdot q \cdot \top}) \cdot \neg {}^{\ulcorner}((\neg q \cdot a)^{*} \cdot \neg(p + q))$$

$= \quad \{\!\![$ by Theorem 6.1.3 and definition of box $]\!\!\}$

$$\mathsf{A}(a^{*} \cdot q \cdot \top) \cdot [(\neg q \cdot a)^{*}](p + q)$$

$= \quad \{\!\![$ by Lemma 4.2 $]\!\!\}$

$$(\mathsf{AF}q) \cdot [(\neg q \cdot a)^{*}](p + q).$$

$\square$

To round off the picture we show the validity of the usual least-fixpoint characterisation of $\mathsf{A}(u)$, where $u = [\![\varphi \mathsf{U} \psi]\!]$ for state formulas φ and ψ. Then, by Lemma 4.1, the definition of f, Lemma 6.4 twice and Corollary 7.2, we obtain

$$\mathsf{A}(u) \;=\; \mathsf{A}(f(u)) \;=\; \mathsf{A}(q \cdot \top + p \cdot a \cdot u) \;=\; q \cdot \top + p \cdot \mathsf{A}(a \cdot u) \;=\; q \cdot \top + p \cdot \mathsf{A}(a \cdot \mathsf{A}(u)).$$

In general quantales, however, $\mathsf{A}(u)$ need not be the least fixpoint of the associated function. We need an additional assumption, namely that unlimited finite iteration can be extended to infinite iteration in the following sense:

$$\forall\, b \in S : \bigsqcap_{i \in \mathbb{N}} {}^{\ulcorner}(b^i) \leq {}^{\ulcorner}(b^\omega)\,. \tag{7}$$

In particular, S must have "enough" infinite elements to make $b^\omega \neq 0$ if all $b^i \neq 0$. This property is e.g. violated in the subquantale LAN of WOR in which only languages of finite words are allowed, because in LAN finite languages may be iterated indefinitely, but no infinite "limits" exist in LAN.

Now we can show the desired leastness of A.

Theorem 7.5 *Assume (7).*

1. $\neg{}^{\ulcorner}(b^\omega) \;=\; \triangle b$.
2. *If* $[\![\varphi]\!] \;=\; p \cdot \top$ *then* $[\![\mathsf{AF}\varphi]\!]_d \;=\; \triangle\neg p \cdot a$.
3. $[\![\varphi \mathsf{U} \psi]\!]_d \;=\; \mu h$, *where* $h(x) \;=_{df}\; q + p \cdot [a]x$.

Proof. 1. First, $\neg{}^{\ulcorner}(b^\omega)$ is a fixpoint of $[b]$:

$$\neg{}^{\ulcorner}(b^\omega) \;=\; \neg{}^{\ulcorner}(b \cdot (b^\omega)) \;=\; \neg{}^{\ulcorner}(b \cdot \neg\neg(b^\omega)) \;=\; [b](\neg{}^{\ulcorner}(b^\omega)).$$

Hence $\triangle b \;=\; \mu[b] \leq \neg{}^{\ulcorner}(b^\omega)$. For the converse inequation we calculate

$$\begin{array}{ll} & \neg{}^{\ulcorner}(b^\omega) \leq \triangle b \\ \Leftrightarrow & \quad \{\![\text{ shunting }]\!\} \\ & \neg\triangle b \leq {}^{\ulcorner}(b^\omega) \\ \Leftarrow & \quad \{\![\text{ by (7) }]\!\} \\ & \neg\triangle b \leq \bigsqcap_{i \in \mathbb{N}} {}^{\ulcorner}(b^i) \\ \Leftarrow & \quad \{\![\text{ definition infimum }]\!\} \\ & \forall\, i \in \mathbb{N} : \neg\triangle b \leq {}^{\ulcorner}(b^i)\,. \end{array}$$

Using $\neg\triangle b \leq 1$, isotony of domain, the definition of box and that $\triangle b$ is a fixpoint of $[b]$, we have indeed

$${}^{\ulcorner}(b^i) \geq {}^{\ulcorner}(b^i \cdot \neg\triangle b) = \neg[b^i]\triangle b = \neg\triangle b.$$

2. Immediate from Theorem 7.4.6 and 1.
3. ¿From the definition of h we get by Boolean algebra

$$h(x) \;=\; (q + p) \cdot (q + [a]x).$$

Now the claim follows from (5), Theorem 7.4.8 and 2. □

8 From CTL* to LTL

The logic LTL is the fragment of CTL* in which only A may occur, outermost, as path quantifier. More precisely, the LTL path formulas are given by

$$\Pi ::= \Phi \mid \bot \mid \Pi \to \Pi \mid \mathsf{X}\,\Pi \mid \Pi \,\mathsf{U}\, \Pi.$$

The LTL semantics is embedded into the CTL* one by assigning to $\varphi \in \Pi$ the semantic value $[\![\mathsf{A}\varphi]\!]$.

Unfortunately, except for the cases $[\![\mathsf{AX}\varphi]\!] = [a][\![\mathsf{A}\varphi]\!]$ and $[\![\mathsf{AG}\varphi]\!] = [a^*][\![\mathsf{A}\varphi]\!]$ the semantics does not propagate nicely in an inductive way into the subformulas, and so a simplified semantics cannot be obtained directly from the CTL* one.

However, by a slight change of view we can still achieve our goal. In the considerations based on the concrete quantales WOR and STR, the semantic element a representing X "glued" transitions to the front of traces. However, as is frequently done, one can also interpret a as a relation that maps a trace σ to its remainder σ^1. This is the basis for a simplified semantics of LTL.

Similarly to before we set

$$[\![\mathsf{X}\varphi]\!]_{\mathsf{LTL}} =_{df} \langle a\rangle[\![\varphi]\!]_{\mathsf{LTL}}.$$

What does axiom (1) mean in this interpretation? It is equivalent to the equation $\langle a\rangle = [a]$ which characterises $\langle a\rangle$ as a total function. This holds indeed for the relation sending σ to σ^1, since standard LTL considers only infinite traces.

What are the tests involved in this case? They have now to be interpreted as sets of paths, since they are subrelations of the identity relation on traces. So in this view the semantics of LTL formulas is again given by test ideals, only in a different algebra.

Therefore we can re-use the simplified CTL semantics. In particular, with $p =_{df} [\![\varphi]\!]_{\mathsf{LTL}}$ and $q =_{df} [\![\psi]\!]_{\mathsf{LTL}}$, we want $[\![\varphi \mathsf{U} \psi]\!]_{\mathsf{LTL}}$ to be the least fixpoint of the function $h(x) =_{df} q + p \cdot \langle a\rangle x$, which by the dual of (5) is $\langle (p \cdot a)^*\rangle q$.

By this, the semantics of $\mathsf{F}\psi$ and $\mathsf{G}\psi$ work out to $\langle a^*\rangle q$ and $[a^*]q$.

Further details are omitted for lack of space.

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