Semiring Neighbours: An Algebraic Embedding and Extension of Neighbourhood Logic

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Abstract. In 1996 Zhou and Hansen proposed a first-order interval logic called *Neighbourhood Logic* (NL) for specifying liveness and fairness of computing systems and also defining notions of real analysis in terms of expanding modalities. After that, Roy and Zhou presented a sound and relatively complete Duration Calculus as an extension of NL.
We present an embedding of NL into an idempotent semiring of intervals. This embedding allows us to extend NL from single intervals to sets of intervals as well as to extend the approach to arbitrary idempotent semirings. We show that most of the required properties follow directly from Galois connections, hence we get the properties for free. As one important result we get that some of the axioms which were postulated for NL can be dropped since they are theorems in our generalisation. At the end of the paper we shortly present some possible applications for neighbours beyond intervals. Here we discuss for example reachability in graphs and applications for hybrid systems.

1 Introduction and related work

Chop-based interval temporal logics, such as ITL [4] and IL [2] are useful for the specification and verification of safety properties of real-time systems. In these logics, one can easily express a lot of properties such as

"if ϕ holds for an interval, then there is a subinterval where ψ holds“.

As it is shown in [13], these logics cannot express all desired properties. E.g., (unbounded) liveness properties such as

"eventually there is an interval where ϕ holds“

is not expressible in these logics. Surprisingly, these logics cannot even express state transitions. That is why in Chapter 9 of [13] extra atomic formulas are introduced. As it is shown there the reason is that the modality *chop* $\frown$, is a *contracting* modality, in the sense that the truth value of $\phi\frown\psi$ on $[b, e]$ only depends on subintervals of $[b, e]$:

$\phi\frown\psi$ holds on $[b, e]$ iff
there exists $m \in [b, e]$ such that ϕ holds on $[b, m]$ and ψ holds on $[m, e]$.

Hence Zhou and Hansen proposed a first-order interval logic called *Neighbourhood Logic* (NL) in 1996 [14]. This first-order logic was proposed for specifying liveness and fairness of computing systems and also defining notions of real analysis in terms of expanding modalities. In 1997 Roy and Zhou presented a sound and relatively complete Duration Calculus as an extension of NL [11]. They had already shown that the basic unary interval modalities of [5] and the three binary interval modalities (C, T and D) of [12] could be defined in NL.

In this paper, we present an embedding of NL into the semiring of intervals presented e.g. in [8]. This embedding allows us to extend NL from single intervals to sets of intervals as well as to extend the approach to arbitrary idempotent semirings. Because of work done in [14] it is also an extension of [5] and [12]. In Section 4 we show that most of the required properties follow directly from Galois connections, hence we get the properties for free. As one important result we get that some of the axioms which were postulated for NL can be dropped since they are theorems in our generalisation. At the end of the paper we briefly present some possible interpretations of neighbours in other models. Here we discuss for example reachability in graphs and applications for hybrid systems. Due to lack of space all proofs are skipped. They can be found in [7].

2 About Neighbourhood Logic

In [14] Zhou and Hansen introduce *left* and *right* neighbourhoods as primitive intervals to define other unary and binary modalities of intervals in a first-order logic. For this, we need intervals as carrier sets. That is why we define *intervals* over a poset of *timepoints* in the usual way as

$$[b,e] \stackrel{\text{def}}{=} \{x : b \leq x \leq e\}\ , \text{ where } b \leq e,$$

$b, e, x \in \mathbb{T}\mathrm{ime}$ and $(\mathbb{T}\mathrm{ime}, +, 0)$ is a monoid. Furthermore, we postulate a subtraction $-$ on $\mathbb{T}\mathrm{ime}$ satisfying for any interval $[b,e]$ the equations $e - b \geq 0$ and $e - b = 0 \Leftrightarrow e = b$. Hence, it is possible to calculate the *length* l of the interval $[b,e]$ as $e - b$. Additionally, $\mathbb{T}\mathrm{ime}$ has to be cancellative w.r.t. $+$. E.g. one can use $\mathbb{R}$, the set of real numbers, as $\mathbb{T}\mathrm{ime}$.
The two proposed simple expanding modalities $\Diamond_l \phi$ and $\Diamond_r \phi$ are defined as follows:

$\Diamond_l \phi$ holds on $[b,e]$ iff there exists $\delta \geq 0$ such that ϕ holds on $[b-\delta, b]$,
$\Diamond_r \phi$ holds on $[b,e]$ iff there exists $\delta \geq 0$ such that ϕ holds on $[e, e+\delta]$,

where ϕ is a *formula* of NL, which is either true or false.[1] With $\Diamond_r(\Diamond_l)$ one can reach the *left* (*right*) *neighbourhood* of the beginning (ending) point of an interval:

$$\overbrace{a \qquad\quad b}^{\phi}\overbrace{\qquad\qquad\qquad\qquad e}^{\Diamond_l \phi} \qquad\qquad \overbrace{b \qquad\qquad\qquad\qquad e}^{\Diamond_r \phi}\overbrace{\qquad\quad c}^{\phi}$$

where $a = b - \delta$ where $c = e + \delta$

[1] The exact definition of the syntax can be found in, e.g., [14, 7].

In contrast to the chop operator the neighbourhood modalities are *expanding* modalities, i.e., they are not contracting operators. Thus $\Diamond_l$ and $\Diamond_r$ depends not only on subintervals of an interval $[b,e]$, but also on intervals "outside". In [14] it is shown that the modalities of [5] and [12] as well as the chop operator can be expressed by the neighbourhood modalities.

3 Embedding Neighbourhood Logic into semirings

First, we repeat the basic definitions of semirings and related algebraic structures and operators. More details about semirings, domain semirings, etc. can be found in [6, 1, 3].

A *semiring* is a quintuple $(S,+,\cdot,0,1)$ such that $(S,+,0)$ is a commutative monoid and $(S,\cdot,1)$ is a monoid such that $\cdot$ is distributive over $+$ and S is *strict*, i.e., $0\cdot a = 0 = a\cdot 0$. The semiring is *idempotent* if $+$ is, i.e., $a+a=a$. On idempotent semirings the relation $a\leq b \overset{\text{def}}{\Leftrightarrow} a+b=b$ is a partial order, called the *natural order* on S. The definition implies that 0 is the least element and $+$ and $\cdot$ are isotone with respect to $\leq$. If S has a greatest element we denote it by $\top$. An important semiring is REL, the algebra of binary relations over a set under relational composition.

A *test semiring* is a pair $(S,\mathsf{test}(S))$, where S is an idempotent semiring and $\mathsf{test}(S)\subseteq[0,1]$ is a Boolean subalgebra of the interval $[0,1]$ of S such that $0,1 \in \mathsf{test}(S)$ and join and meet in $\mathsf{test}(S)$ coincide with $+$ and $\cdot$. This definition corresponds to the one in [10]. We will use $a,b,c\ldots$ and x,y,z for arbitrary S-elements and $p,q,r,\ldots$ for tests. By $\neg$ we denote complementation in $\mathsf{test}(S)$; implication $p\rightarrow q = \neg p + q$ obeys its standard laws.

A *domain semiring* is a pair $(S,\ulcorner)$, where S is a test semiring and the *domain* operation $\ulcorner\colon S\rightarrow\mathsf{test}(S)$ satisfies

$$a\leq {\ulcorner a}\cdot a \quad \text{(d1)}, \qquad {\ulcorner(p\cdot a)}\leq p \quad \text{(d2)}.$$

The relevant consequences are shown in [1]. In particular, domain is universally disjunctive and hence $\ulcorner$ is strict, i.e., $\ulcorner 0 = 0$. Furthermore we can expand (d1) to the equation $a = {\ulcorner a}\cdot a$ (d1'). A corresponding codomain operation $\urcorner : S\rightarrow \mathsf{test}(S)$ can defined analogously. S is called a *bidomain semiring* if there are domain and codomain operations. In bidomain semirings we have the following separability property:

$$a^{\urcorner}\cdot{\ulcorner b}\leq 0 \Leftrightarrow a^{\urcorner}\cdot b\leq 0 \Leftrightarrow a\cdot{\ulcorner b}\leq 0\ . \tag{sep}$$

In [8] we showed that the structure INT $=(\mathcal{P}(\mathbb{I}),\cup,;,\emptyset,\mathbb{1})$ is an idempotent semiring, where $\mathbb{I}\overset{\text{def}}{=}\{[b,e] : b\leq e, b,e\in\mathbb{T}\mathrm{ime}\}$ is the set of all intervals, $; : \mathcal{P}(\mathbb{I})\times\mathcal{P}(\mathbb{I})\rightarrow\mathcal{P}(\mathbb{I})$ defines the elementwise interval composition and $\mathbb{1}\overset{\text{def}}{=}\{[b,b] : b\in\mathbb{T}\mathrm{ime}\}$ is the neutral element with respect to multiplication. The definition of interval composition says that $[a,b]\,;[c,d]$ is defined if and only if $b=c$, i.e., iff the interval $[c,d]$ is part of the "right neighbourhood" of $[a,b]$, or, symmetrically, iff $[a,b]$ is part of the "left neighbourhood" of $[c,d]$. Here the domain (codomain) characterises the starting points (end points) of intervals, i.e., for $x \in \mathcal{P}(\mathbb{I})$

$$ {}^\ulcorner x \;=\; \{[b,b] \;:\; [b,e] \in x\} \qquad \text{and} \qquad x^\urcorner \;=\; \{[e,e] \;:\; [b,e] \in x\} \;. $$

These operators imply the following view of the neighbourhood modalities.

$$ \begin{aligned} \Diamond_r \phi \text{ holds on } \{[b,e]\} &\Leftrightarrow \exists\, [u_1,u_2] \in \mathbb{I}_\phi : [b,e]\,;[u_1,u_2] \text{ is defined} \\ &\Leftrightarrow \{[b,e]\}^\urcorner \le {}^\ulcorner(\mathbb{I}_\phi), \end{aligned} $$

In general we have for $\Diamond_l \phi$ and $\Diamond_r \phi$ the following equivalences:

$$ \begin{gathered} \Diamond_l \phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \text{ iff } {}^\ulcorner x \le \mathbb{I}_\phi{}^\urcorner \;, \\ \Diamond_r \phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \text{ iff } x^\urcorner \le {}^\ulcorner\, \mathbb{I}_\phi \;. \end{gathered} $$

As a first result we show that at least one of the eight axioms, which are postulated in [14] can be dropped and is in fact a theorem in bidomain semirings. More simplifications on calculations are given in Section 4 after introducing a general form of neighbourhoods.

Lemma 3.1 *$\Diamond(\phi \vee \psi) \Leftrightarrow \Diamond\phi \vee \Diamond\psi$, where $\Diamond$ is $\Diamond_r$ or $\Diamond_l$. Hence Axiom 4 of [14] is a conclusion.*

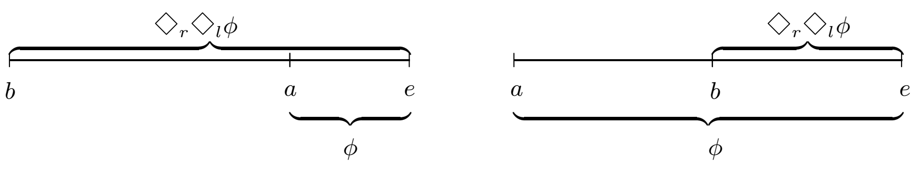
Now we will discuss the box operators $\Box_l \phi \stackrel{\text{def}}{=} \sim\Diamond_l{\sim}\phi$ and $\Box_r \stackrel{\text{def}}{=} \sim\Diamond_l{\sim}\phi$ of Zhou and Hansen in detachment and bidomain semirings, respectively. Here, $\sim$ is the negation of truth values, i.e., $\sim(\mathsf{true}) = \mathsf{false}$ and $\sim(\mathsf{false}) = \mathsf{true}$. In [13, 14] it is denoted as usual by $\neg$. But this symbol clashes with the negation symbol of tests. The meaning of $\Box_l \phi$ ($\Box_r \phi$) is:

$$ \Box_l \phi (\Box_r \phi) \text{ holds on } [b,e] \Leftrightarrow \phi \text{ holds on all neighbours left (right) of } [b,e] \;. $$

In bidomain semirings we get a generalised form by:

$$ \begin{gathered} \Box_l \phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \text{ iff } (\overline{\mathbb{I}_\phi})^\urcorner \,;\, {}^\ulcorner x \le 0 \qquad \text{and} \\ \Box_r \phi \text{ holds on } x \in \mathcal{P}(\mathbb{I}) \text{ iff } x^\urcorner \,;\, {}^\ulcorner(\overline{\mathbb{I}_\phi}) \le 0 \;. \end{gathered} $$

In [14] the authors introduce the composed neighbourhood modalities $\Diamond_r \Diamond_l \phi$ and $\Diamond_l \Diamond_r \phi$ and called them *converses*. But these are very unhandy in calculations and we show that they are again diamonds closely related to $\Diamond_l$ and $\Diamond_r$. We want to illustrate the meaning of $\Diamond_r \Diamond_l \phi$. Here, we have that either $[a,e]$ is a postfix of $[b,e]$ or, if $a \le b$, $[b,e]$ is a postfix of $[a,e]$:



where $a = e - \delta$.

Now we have a look at $\Diamond_r\Diamond_l\phi$ using domain and codomain.

$$\begin{aligned}
\Diamond_r\Diamond_l\phi \text{ holds on } x &\Leftrightarrow x\urcorner \le \ulcorner(\mathbb{I}_{\Diamond_l\phi}) \\
&\Leftrightarrow x\urcorner \le \ulcorner\{[b,e] : \ulcorner\{[b,e]\} \le \mathbb{I}_\phi\urcorner\} \\
&\Leftrightarrow x\urcorner \le \mathbb{I}_\phi\urcorner \ , \\
\Diamond_l\Diamond_r\phi \text{ holds on } x &\Leftrightarrow \ulcorner x \le \ulcorner\mathbb{I}_\phi \ .
\end{aligned}$$

We see that $\Diamond_r\Diamond_l\phi$ and $\Diamond_l\Diamond_r\phi$ can be as easily expressed as the single diamonds introduced above. The four neighbourhood operators $(\Diamond_l, \Diamond_r, \Diamond_l\Diamond_r, \Diamond_r\Diamond_l)$ represent all combinations for comparing domain and codomain and therefore motivate the generalised definition in the next section.

4 Generalised Neighbourhoods and some Properties

Starting with the definitions of neighbourhoods given in Section 3 and motivated by NL we give general definitions, which work on bidomain semirings.

Definition 4.1 Let S be a bidomain semiring and $x, y \in S$. Then
(i) x is a *left neighbour* of y (or $x \le \overset{n}{\Diamond}_l y$ for short) iff $x\urcorner \le \ulcorner y$,
(ii) x is a *right neighbour* of y (or $x \le \overset{n}{\Diamond}_r y$ for short) iff $\ulcorner x \le y\urcorner$,
(iii) x is a *left boundary* of y (or $x \le \overset{b}{\Diamond}_l y$ for short) iff $\ulcorner x \le \ulcorner y$,
(iv) x is a *right boundary* of y (or $x \le \overset{b}{\Diamond}_r y$ for short) iff $x\urcorner \le y\urcorner$.

We will see below that the notation using $\le$ is justified. Now we have a closer look at the definition and its interpretation in INT. For example (i) describes the situation, where for each element $[a, b]$ of x there exists at least one interval in y with starting point b. Hence $\Diamond_r\phi$ holds on x if and only if x is a left neighbour of $\mathbb{I}_\phi$ ($x \le \overset{n}{\Diamond}_l\mathbb{I}_\phi$). The change in direction (left, right) follows from the point of view. $\Diamond_r\phi$ starts with an interval of x and has a look at elements of $\mathbb{I}_\phi$ at its right, whereas our definitions start at $\mathbb{I}_\phi$. Starting at our definitions of neighbours and borders we calculate an explicite form of these operations.

Lemma 4.2 *Neighbours and boundaries can be expressed explicitly by*

$$\begin{aligned}
\overset{n}{\Diamond}_l y &= \top\cdot\ulcorner y \ , \qquad & \overset{n}{\Diamond}_r y &= y\urcorner\cdot\top \ , \\
\overset{b}{\Diamond}_l y &= \ulcorner y\cdot\top \ , \qquad & \overset{b}{\Diamond}_r y &= \top\cdot y\urcorner \ .
\end{aligned}$$

If there is a complementation function $\overline{}$ on S, which satisfies $\overline{\overline{a}} = a$, $\overline{a} + a = \top$ and $a \le b \Leftrightarrow \overline{b} \le \overline{a}$, we define perfect neighbours and perfect boundaries.

Definition 4.3 Let S be a complement bidomain semiring and $x, y \in S$.
(i) x is a *perfect left neighbour* of y (or $x \le \overset{n}{\Box}_l y$ for short) iff $x\urcorner\cdot\ulcorner\overline{y} \le 0$,
(ii) x is a *perfect right neighbour* of y (or $x \le \overset{n}{\Box}_r y$ for short) iff $\overline{y}\urcorner\cdot\ulcorner x \le 0$,
(iii) x is a *perfect left boundary* of y (or $x \le \overset{b}{\Box}_l y$ for short) iff $\ulcorner x\cdot\ulcorner\overline{y} \le 0$,
(iv) x is a *perfect right boundary* of y (or $x \le \overset{b}{\Box}_r y$ for short) iff $x\urcorner\cdot\overline{y}\urcorner \le 0$.

By (iii) and (iv) we have an additional extension of NL. These two definitions define "box-operators" for the converses of neighbourhood modalities, which are not defined in the semantics of NL given in [13]. To justify the definitions above we have

Lemma 4.4 *Each perfect neighbour (boundary) is a neighbour (boundary).*

We can characterise the box operations, like neighbours/boundaries, in an explicit form.

Lemma 4.5 *Perfect neighbours and perfect boundaries have the following explicit forms:*

$$\boxed{\scriptstyle\eta}_l y \;=\; \top \cdot \neg^{\ulcorner}(\overline{y}) \;, \qquad \boxed{\scriptstyle\eta}_r y \;=\; \neg(\overline{y})^{\urcorner} \cdot \top \;,$$
$$\boxed{\scriptstyle\beta}_l y \;=\; \neg^{\ulcorner}(\overline{y}) \cdot \top \;, \qquad \boxed{\scriptstyle\beta}_r y \;=\; \top \cdot \neg(\overline{y})^{\urcorner} \;.$$

To reduce calculations we introduce $\Diamond$ and $\boxdot$ as parameters, which can be instantiated by either $\langle\eta\rangle_l$, $\langle\eta\rangle_r$, $\langle\beta\rangle_l$ or $\langle\beta\rangle_r$ and $\boxed{\scriptstyle\eta}_l$, $\boxed{\scriptstyle\eta}_r$, $\boxed{\scriptstyle\beta}_l$ or $\boxed{\scriptstyle\beta}_r$, respectively. If the "direction" of $\Diamond$ or $\boxdot$ is important we use formulas like $\Diamond_l$ and $\boxdot_r$ where only one degree of freedom remains.
Boxes and diamonds are connected via the de Morgan duality

$$\boxdot y \;=\; \overline{\Diamond\overline{y}},$$

hence they form proper modal operations. Additionally, it follows that diamonds and boxes are lower and upper adjoints of Galois connections:

$$\Diamond_l x \le y \;\Leftrightarrow\; x \le \boxdot_r y \;, \qquad \Diamond_r x \le y \;\Leftrightarrow\; x \le \boxdot_l y \;.$$

By the Galois connections and de Morgan dualities we get many properties of (perfect) neighbours and (perfect) boundaries for free. For example we have, with $x \sqcap y \;=\; \overline{\overline{x} + \overline{y}}$,

Corollary 4.6

(i) *$\Diamond$ and $\boxdot$ are isotone.*
(ii) *$\Diamond$ is distributive and $\boxdot$ is conjunctive,*
i.e., $\Diamond(x + y) \;=\; \Diamond x + \Diamond y$ and $\boxdot(x \;\sqcap\; y) \;=\; \boxdot x \;\sqcap\; \boxdot y$.
(iii) *We also have the cancellative laws*
$\Diamond_l \boxdot_r x \le x \le \boxdot_r \Diamond_l x$ and $\Diamond_r \boxdot_l x \le x \le \boxdot_l \Diamond_r x$.

In sum, all theorems given in [13, 11, 14] hold in the generalisation, too. Most of them are already proved by the Galois connection and the Corollary above.

Lemma 4.7

(i) $\langle\eta\rangle_l \langle\eta\rangle_r y \;=\; \langle\eta\rangle_l y$ *and* $\langle\eta\rangle_r \langle\eta\rangle_l y \;=\; \langle\eta\rangle_r y$.
(ii) $\langle\eta\rangle_l \langle\eta\rangle_r y \le \boxed{\scriptstyle\eta}_l \langle\eta\rangle_r y$ *and* $\langle\eta\rangle_r \langle\eta\rangle_l y \le \boxed{\scriptstyle\eta}_r \langle\eta\rangle_l y$.
(iii) $\boxed{\scriptstyle\eta}_l \boxed{\scriptstyle\eta}_r y \;=\; \boxed{\scriptstyle\eta}_r y$ *and* $\boxed{\scriptstyle\eta}_r \boxed{\scriptstyle\eta}_l y \;=\; \boxed{\scriptstyle\eta}_l y$

Lemma 4.7.(ii) is the same as Axiom 6 of [14], which is now a theorem. There are many more simplifications and extensions for NL which we do not discuss here.

Interpretation in other models

We generalised NL to arbitrary bidomain semirings. Thus we are able to adopt the theory to other areas. Bidomain semirings having applications in computer science are for example

- REL, the algebra of binary relations over a set under relational composition,
- LAN, the algebra of formal languages under language concatenation, and
- PAT, the algebra of sets of graph paths under path fusion.

More semirings and applications can be found e.g. in [1, 6]. In all these semirings we can interpret (perfect) neighbours and (perfect) boundaries. In PAT for example, $\Diamond^{\eta}_{r}T$ is the set of all paths which can be reached from the paths in T. In contrast to this, $\Box^{\eta}_{r}T$ is the set of all paths which can only be reached from T.

In a discrete semiring S, i.e., $\mathsf{test}(S) = \{0,1\}$, like LAN, all diamonds ($\Diamond^{\eta}_{l}$, $\Diamond^{\eta}_{r}$, $\Diamond^{\beta}_{l}$, $\Diamond^{\beta}_{r}$) are the same and all boxes collapses, too. We have

$$\Diamond^{\cdot} L = \begin{cases} 0 \text{ if } L = \emptyset \\ \top \text{ otherwise,} \end{cases} \qquad \boxdot L = \begin{cases} \top \text{ if } L = \top \\ 0 \text{ otherwise.} \end{cases}$$

Based on INT we presented an embedding of the Duration Calculus in idempotent semirings in [8]. Here we can adopt the theory, too. In [7, 9] we introduced an algebra of processes, where processes are sets of trajectories. These models are a first step towards the description of hybrid systems in an algebraic manner. The right neighbour $\Diamond^{\eta}_{r}$ characterises properties of trajectories which will be reached in the future. More informations concerning more details about the interpretations of neighbours/boundaries as well as interpretations in other models can be found in [7].

5 Conclusion and Outlook

In this paper we started with the Neighbourhood Logic developed by Zhou and Hansen. We showed that we can embed NL into the theory of semirings. With the help of the embedding we showed that at least two axioms can be dropped in the definition of NL and that neighbours can be expressed in a much more general framework. Therefore we presented neighbours and boundaries in bidomain semirings and presented important Galois connections. At the end we gave a short discussion for further applications of the generalised version of NL.

Möller developed the theory of Lazy semirings and we presented an algebra for hybrid systems in [9]. Thus we want to adapt and, if necessary, modify the neighbours and boundaries to Lazy semirings. Then we have a further application for NL in a theory where we can express unlimited processes.

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