An Algebraic Semantics for Duration Calculus

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ABSTRACT. We present an algebraic semantics for Duration Calculus based on semirings and quantales. Duration Calculus was originally introduced in 1991 as a powerful logic for specifying the safety of real-time systems. We embed the Duration Calculus into the theory of Boolean semirings and extend them to Kleene algebras and ω-algebras, respectively, to express finite and infinite iteration. This allows us to calculate easily with the safety requirements and to see special results of the Duration Calculus in a more general context. When formulating an algebraic semantics we also generalise parts of von Karger's work about reactive systems, especially, the engineer's induction.

1 Introduction

Reactive systems are systems that interact with their environment on an ongoing, nearly never-ending basis. For example, a traffic management system has no particular point in time in which it has all its inputs ready. Rather, inputs keep on arriving, from various entities in its environment as time progresses and the system, whether hardware or software, is repeatedly invoked. A special class of reactive systems are the *real-time systems*.

In 1991 Zhou, Hoare and Ravn have developed the *Duration Calculus* (DC) [16] for specifying all kinds of requirements of reactive and, especially, real-time systems. These specifications include functional requirements as well as dependability requirements. To support the verification and the design of reactive systems the language of DC has been developed further to be able to describe systems with more details [4].

This paper is based on and extends the work about approaches to reactive systems by von Karger [13]. Especially, we generalise the engineer's induction, which is a useful theorem for calculations concerning finite iteration. Also, this paper is based on work by Hansen and Zhou [3] as well as by the research

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group at UNU/IIST with their project *Design Techniques for Real-Time Systems* (e.g. [17]).

We present an algebraic semantics for DC. The main advantage of an algebraic characterisation is to avoid complicated formulas with many temporal operators like the ∃- and the ∀-operator. This offers the possibility to formulate safety requirements in a short and elegant way. Doing this, we can calculate very easily with the safety specifications and other requirements coming from DC. When calculating the requirements, we define multimodal operators, which generalise the temporal ones of von Karger. This allows us to embed his work in a more general context and to extend his results.

For our algebraic approach we use the well-known structure of semirings (e.g. [5]) and Kleene algebras (e.g. [8, 2]), because they already provide a good basis for many applications as for example the description of imperative computer programs and modal logics like linear temporal logic. Therefore these algebraic structures offer a promising access to an algebraic semantics.

The paper is organised in the following way. In section 2 we give a description of the famous gas burner example. It was given by Zhou et al. [16] when introducing DC and can be seen as the first short application example of DC. In the sections 3 we recapitulate the definitions of the used algebraic structures, like semirings. Section 4 establishes a basis to embed this example into the theory of semirings. Here we define some modal operators. The embedding is done in section 5. Finally we present an algebraic approach to DC and extend the theory to infinite iteration using an ω-operator.

2 An Interval-Based Model for Duration Calculus

We start with recapitulating the classical example of a leaking gas burner, which was introduced in [10]. In [3] the problem is given as follows:
'A gas burner is either heating when the flame is burning or idling when the flame is not burning. Usually, no gas is flowing while it is idling. However, when changing from idling to heating, gas must be flowing for a short time before it can be ignited, and when a flame failure appears, gas must be flowing before the failure is detected and the gas valve is closed.'
Obviously, there can exist some time where the flame is not burning but gas is flowing. In that case gas is *leaking*.
A design of a safe gas burner must guarantee that the time intervals with leaking gas do not take too long. Let us assume the following safety requirement for a leaking gas pipe.

’For any observation interval that is shorter than 30 seconds, the accumulation of leakage must be less than 4 seconds.’ (Req)

In the sequel we develop an algebraic expression for this safety specification. Obviously, we need some algebraic structure on intervals, binary operations on intervals to concatenate them and some measure for the leakage.
An *interval* of an ordered set $(M, \leq)$ is defined as

$$[a,b] \stackrel{\text{def}}{=} \{x : x \in M,\ a \leq x \leq b\}$$

for any $a, b \in M$ with $a \leq b$. If $(M, +, 0)$ forms a group, the *duration* of an interval $[a,b]$ is defined by $b - a$. We define the composition of two intervals $[a,b]$ and $[c,d]$ $(a,b,c,d \in M)$ as

$$[a,b] \,;\, [c,d] \stackrel{\text{def}}{=} \begin{cases} [a,d] & \text{if } b = c \\ \text{undefined} & \text{otherwise .} \end{cases}$$

M may include the special element ∞, like $\mathbb{R} \cup \{\infty\}$. If so, intervals $[a,\infty]$ are required to be *left-annihilators* w.r.t. composition, i.e., $[a,\infty] \,;\, [c,d] = [a,\infty]$ for any interval $[c,d]$. Hence ∞ is the greatest element in M and a third case has to add to the interval composition which covers the case $b = \infty$. Assuming M is a subset of $\mathbb{R} \cup \{\infty\}$ and $\infty \in M$, $[a,\infty]$ can be considered as the well-known, left open interval $[a,\infty)$. Extending ; to sets of intervals by

$$U \,;\, V \stackrel{\text{def}}{=} \{u \,;\, v \ :\ u \in U,\ v \in V,\ u \,;\, v \text{ defined}\},$$

where $U, V \in \mathcal{P}(\text{Int})$ and Int describes the set of all intervals over a set M, we can consider the structure $\text{INT} \stackrel{\text{def}}{=} (\mathcal{P}(\text{Int}), \cup, \emptyset, ;, 1\!\!1_{\text{Int}})$. Here $1\!\!1_{\text{Int}} \stackrel{\text{def}}{=} \{[a,a] \ :\ a \in M\}$ is the identity element with respect to ;.

In the sequel we restrict our example to be a system with real numbers $\mathbb{R}$ as the set of time M. Let $\chi : \mathbb{R} \to \{0,1\}$ be a Boolean function that is Lebesgue integrable. $\chi(t) = 1$ says that the pipe leaks at time t, and $\chi(t) = 0$ otherwise. Since Lebesgue-integrable implies Riemann-integrable, we can use nearly arbitrary functions for χ. Now, we can use the Lebesgue integral for measuring the leakage in an interval $[a,b]$, if $a, b \in \mathbb{R}$.

$$\begin{array}{rcl} leak : \text{Int} & \to & \mathbb{R} \cup \{\infty\} \\ [a,b] & \mapsto & \int_a^b \chi(t)\, dt \end{array}$$

With these definitions, we return to the safety property (Req) of the beginning. This can be now reformulated as the implication

$$\forall [a,b] \in \text{Int} : \; b - a \leq 30 \;\Rightarrow\; \mathit{leak}([a,b]) \leq 4 \;. \qquad \text{(Req1)}$$

Before we reformulate this requirement again, we will first take a closer look at the algebraic structure of INT.

3 Basic definitions and algebraic structures

A *semiring* is a quintuple $(S, +, 0, \cdot, 1)$ such that $(S, +, 0)$ is a commutative monoid and $(S, \cdot, 1)$ is a monoid such that $\cdot$ distributes over $+$ and 0 is an annihilator, i.e., $0 \cdot a = 0 = a \cdot 0$. A semiring is called *idempotent* if $+$ is idempotent. Then the *natural order* $\leq$ on S is given by $a \leq b \overset{\text{def}}{\Leftrightarrow} a + b = b$ for $a, b \in S$. In an idempotent semiring both operations $+$ and $\cdot$ are isotone. Moreover, 0 is the $\leq$-least element with respect to the natural order.

An idempotent semiring S is called a *quantale* if S is a complete lattice under the natural order and $\cdot$ is universally disjunctive in both arguments. Following [1] one might call a quantale also a *standard Kleene algebra*. A *Boolean quantale* is a quantale in which the underlying lattice is a completely distributive Boolean algebra.

Important representatives of idempotent semirings are REL, the algebra of binary relations over a set under relational composition, LAN, the algebra of formal languages under concatenation and PAT, the algebra of path-sets of a given graphs under path fusion. These semirings are even Boolean quantales. More details can be found for example in [2].

We can extend an idempotent semiring by finite and infinite iteration. So, a *Kleene algebra* is a structure $(S, {}^*)$ consisting of an idempotent semiring S and an operation * that satisfies the *unfold* and *induction* axioms, for $a, b, c \in S$

$$\begin{aligned} 1 + a \cdot a^* \leq a^* \;, &\qquad 1 + a^* \cdot a \leq a^* \;, \\ b + a \cdot c \leq c \Rightarrow a^* \cdot b \leq c \;, &\qquad b + c \cdot a \leq c \Rightarrow b \cdot a^* \leq c \;. \end{aligned}$$

For example PAT becomes a Kleene algebra by defining V^* as $\bigcup_{n \in \mathbb{N}} V^n$ for paths-sets V. To express infinite iteration we axiomatise an ω-operator over a Kleene algebra. A *ω-algebra* is a pair $(S, {}^\omega)$ such that S is a Kleene algebra and ${}^\omega$ satisfies, for $a, b, c \in S$ the *unfold* and *coinduction* axioms

$$a^\omega = a \cdot a^\omega \;, \qquad c \leq a \cdot c + b \Rightarrow c \leq a^\omega + a^* \cdot b \;.$$

We can not only show that PAT, LAN and REL form Kleene algebras and ω-algebras but also give an result for arbitrary quantales. Following [2] we get

Lemma 3.1

1. *Every quantale can be extended to a Kleene algebra by the definition* $a^* \stackrel{\text{def}}{=} \mu x \,.\, a \cdot x + 1$.

2. *If the quantale is even a completely distributive lattice then it can be extended to an ω-algebra by setting* $a^\omega \stackrel{\text{def}}{=} \nu x \,.\, a \cdot x$. *In this case,*

$$\nu x \,.\; a \cdot x + b \;=\; a^\omega + a^* \cdot b \;.$$

The proof is by fixpoint fusion.

Checking all the laws given above for the set of intervals we get the following

Lemma 3.2 *The structure* $\text{INT} = (\mathcal{P}(\text{Int}), \cup, \emptyset, ;, \mathbb{1}_{\text{Int}})$ *forms a Boolean quantale with the subset-relation as the natural order.*

By this, we now have a first basis for an algebraic characterisation. We also have finite iteration * and infinite iteration $^\omega$ with all their laws available.

4 Modal operators

Starting from DC developed by Zhou, Hoare and Ravn [16] it is obvious that we need some modal operators to specify all kinds of requirements of real-time systems, like 'for any interval'. The aim is to avoid the bothering operators, like $\forall$ and $\exists$. In the sequel we show how to achieve this in Boolean quantales. In these, the existence of residuals and detachments is guaranteed. In any quantale the *right residual* a/b and the *left residual* $a\backslash b$ are defined by Galois Connections as

$$x \leq a/b \stackrel{\text{def}}{\Leftrightarrow} x \cdot b \leq a \quad \text{and} \quad x \leq a\backslash b \stackrel{\text{def}}{\Leftrightarrow} a \cdot x \leq b \;.$$

In the Boolean quantale INT, these operations are characterised pointwise by $i \in U/V \Leftrightarrow \forall v \in V : i \,;\, v \in U$ (provided $i \,;\, v$ is defined). Based on residuals, the *right detachment* $a\lfloor b$ and the *left detachment* $a\rfloor b$ are defined as

$$a\lfloor b \stackrel{\text{def}}{=} \overline{\overline{a}/b} \quad \text{and} \quad a\rfloor b \stackrel{\text{def}}{=} \overline{a\backslash\overline{b}} \;.$$

Over intervals these operators are also characterised pointwise and are lifted to INT similarly to $;$. More precisely, $i \in U\lfloor V \Leftrightarrow \exists\, v \in V : i \,;\, v \in U$. Abstractly, the right detachment $a\lfloor b$ is the inverse image of a under $\cdot b$; hence it is suggestive to view it as a forward modal operator. Equivalently, the left detachment and the right residual propose backward operators.

Lemma 4.1 *Setting* $|a\rangle b \stackrel{\text{def}}{=} b\lfloor a$, $\langle a|b \stackrel{\text{def}}{=} b\rfloor a$, $|a]b \stackrel{\text{def}}{=} \overline{|a\rangle\overline{b}}$ *and* $[a|b \stackrel{\text{def}}{=} \overline{\langle a|\overline{b}}$ *converts the residuals and the detachments into proper modal operators:*

1. $|a\rangle, \langle a|$ *are universally disjunctive and* $|a], [a|$ *are universally conjunctive.*
2. $|a \cdot b\rangle c = |a\rangle(|b\rangle c)$ *and* $|a \cdot b]c = |a](|b]c)$. *(The dual laws for the backward operators are also true.)*
3. *For any* b, *the equivalences*
   $$|x\rangle b \le y \Leftrightarrow x \le [y|b \quad \textit{and} \quad \langle x|b \le y \Leftrightarrow x \le [y|b$$
   form two Galois connections.
4. *Differing from general modal operators,*
   $$|a\rangle\langle b|c = \langle b||a\rangle c \quad \textit{and} \quad |a][b|c = [b||a]c \ . \quad \textit{(compatibility)}$$

If we want to describe processes and systems with infinite elements, it is necessary to relax the axioms of the semiring and abandon the right-strictness $a \cdot 0 = 0$. More details about the so-called *left quantales*, the resulting *left Kleene algebra* and *left* ω*-algebra* are described in [9] and [7]. Upon closer examination of Boolean left quantales, it turns out that the right residuals and the right detachment $a\lfloor b$ only exist if b is an element corresponding to a finite interval. Since we want to use the right detachments in general, we restrict ourselves to 'normal' quantales.

As a special case of the compatibility rule of 4.1 we define further modal operators as

$$\langle a\rangle b \stackrel{\text{def}}{=} |a\rangle\langle a|b \ , \qquad [a]b \stackrel{\text{def}}{=} \overline{\langle a\rangle\overline{b}} = |a][a|b \ .$$

Thus $\langle a\rangle b$ ($[a]b$ respectively) holds of x if b holds for at least one extension (for all extensions) of x in a. In von Karger's work [12, 14] the *negative modalities* are a special cases of these combined modal operations, i.e., $\diamondminus b \stackrel{\text{def}}{=} \langle\top\rangle b$ and $\boxminus b \stackrel{\text{def}}{=} [\top]b$, where $\top$ represents the greatest element of the Boolean quantale. The *positive modalities* of von Karger can be interpreted in Boolean semirings (quantales) as $\diamondplus b \stackrel{\text{def}}{=} \top\cdot b\cdot\top$ and $\boxplus b \stackrel{\text{def}}{=} \overline{\diamondplus\overline{b}}$ and can be generalised by

$$\langle a\rangle_+ b \stackrel{\text{def}}{=} a \cdot b \cdot a \ , \qquad [a]_+ b \stackrel{\text{def}}{=} \overline{\langle a\rangle_+\overline{b}} \ .$$

Lemma 4.2 *The modal operators* $\langle a\rangle$, $\langle a\rangle_+$ *and* $[a]$, $\langle a\rangle_+$ *are the lower and upper adjoints of Galois connections. That is*

$$\langle a\rangle b \le c \Leftrightarrow b \le [a]_+ c \quad \textit{and} \quad \langle a\rangle_+ b \le c \Leftrightarrow b \le [a]c \ .$$

As a consequence all these operators are isotone, the lower adjoints ($\langle a\rangle, \langle a\rangle_+$) are universally disjunctive and the upper adjoints are universally conjunctive. Furthermore, we have the cancellation laws

$$\langle a\rangle_+[a]b \le b \le [a]_+\langle a\rangle b \quad \text{and} \quad \langle a\rangle[a]_+b \le b \le [a]\langle a\rangle_+ b \ .$$

Lemma 4.3

1. $\langle a\rangle 0 \;=\; \langle a\rangle_+ 0 \;=\; 0 \quad [a]0 \;=\; [a]_+ 0 \;=\; \top$
2. *If* $1 \ge a$*, then* $[a]_+ b \le b \le \langle a\rangle_+ b$ *and* $[a]b \le b \le \langle a\rangle b$.
   If $1 \le a$*, then* $\langle a\rangle_+ b \le b \le [a]_+ b$ *and* $\langle a\rangle b \le b \le [a]b$.
   Especially, $\langle 1\rangle_+ b \;=\; \langle 1\rangle b \;=\; [1]_+ b \;=\; [1]b \;=\; b$.

Often we need to know how to handle products with these modal operators, like $\langle a\rangle(b\cdot c)$. Generally, we cannot calculate such terms. Thus an important assumption for most calculations in the (extended) Duration Calculus is local linearity

$$\begin{aligned} (a\cdot b)\lfloor c \;&=\; a\cdot b\lfloor c + a\lfloor(c\lfloor b) \\ c\rfloor(a\cdot b) \;&=\; c\rfloor a\cdot b + (a\rfloor c)\rfloor b \ . \end{aligned}$$

These laws describe the cases that occur when c is cut off the right and the left side, respectively, of $a\cdot b$. These can be visualised by the following figure:

In INT the second equation describes the relation $\forall\ x, y : (\exists\ z : x \sqsubseteq z \wedge y \sqsubseteq z) \ \Rightarrow\ x \sqsubseteq y \vee y \sqsubseteq x$, where $\sqsubseteq$ is the prefix-order. A similar situation using the postfix-order is described by the first equation. In combination with DC these axioms can be assumed to be true without much loss of generality, because DC is based on linear temporal logic.
The semiring of the binary relations INT as well as LAN, REL and PAT are locally linear.

5 Duration Calculus

Safety requirements are best described by postulating that certain situations, e.g. explosions or system crashes, may never occur. Remembering the example of the beginning we required that there is no interval that has a duration less or equal than 30 seconds and a total leakage duration greater than 4.

With the help of the quantale INT with the time set M as real numbers we now formulate the safety criterion (Req1) by:

$$\texttt{gas-req} = [\top]_+\overline{s}, \text{ where } s = \{[a,b] \ : \ b - a \leq 30, \, leak([a,b]) > 4\}$$

In [13] it is shown how we can create a possible and safe design of the gas burner by setting:

$$\texttt{gas-design} = t^*, \text{ where } t = \{[a,b] \ : \ b - a = 30, \, leak([a,b]) < 2\}$$

Here we see one main argument to use an algebraic approach. We now have two simple expressions, $[\top]_+\overline{s}$ and t^*. They are easy to handle and we can calculate in an easy and elegant way when proving properties of safety requirements and designs. While doing so, we only have to work with the few axioms of Kleene algebra and can make use of all the knowledge about this algebraic structure.

`gas-design` has the advantage over `gas-req` to include only the intervals with duration of exactly 30 seconds and can be controlled by a looping program. To show the correctness and the safety of the chosen design, we have to achieve the

Lemma 5.1 `gas-design` *is a subset of* `gas-req`.

Proof: Due to the fact that INT is locally linear, we can take advantage of the following so called engineer's induction law for locally linear Boolean Kleene Algebra.

$$1 + a + a \cdot a \leq [\top]_+\overline{b} \ \Rightarrow \ a^* \leq [\top]_+\overline{b}, \text{ if } b \leq [\top]_+\overline{a}$$

The proof for this induction rule takes advantage of $a^* = (\mu_y : 1 + a \cdot y)$ and fixpoint-fusion. The remaining calculations are lengthly, but straightforward.[1] □

We want to present a generalisation of the engineer's induction law for contractive elements. An element c is called *contractive*, if $c \cdot x \leq c$ and $x \cdot c \leq c$ for all x.

Theorem 5.2 *If $c, \overline{c}$ are contractive and $1 \leq c$, then:*

$$b \leq [c]_+\overline{a} \textit{ and } 1 + a + a \cdot a \leq [c]_+\overline{b} \Rightarrow a^* \leq [c]_+\overline{b}$$

The proof is a generalisation of the above induction law and uses the following lemma:

[1] The complete proof can be found in [6]

Lemma 5.3 *Let S be an Boolean quantale and $a, c \in S$.*

1. *If $\overline{c}$ is contractive, then:*

$$\begin{array}{l} 1\lfloor c + a^* \cdot (a\lfloor c) \leq a^*\lfloor c \\ c\rfloor 1 + (c\rfloor a) \cdot a^* \leq c\rfloor a^* \ . \end{array}$$

2. *If $c, \overline{c}$ are contractive and $1 \leq c$, then:*

$$\langle c\rangle a^* \leq \langle c\rangle(1 + a + a \cdot a) + \langle c\rangle_+ a \ .$$

This lemma makes use of fixpoint calculations as well as fixpoint fusion again.

Finally, we give a short excursus about using ω-algebra. We want to model a design for the gas burner with an infinite number of iterations instead of a finite number. This is the case, if we express a loop that never ends, which can be very useful for many applications. For this, we modify `gas-design` in the following equation:

$$\omega\texttt{-gas-design} \ = \ s^\omega, \ \text{where } s = \{[a, b] \ : \ b - a = 30, leak([a, b]) < 2\}$$

In this case we have again the advantage of an algebraic form. Furthermore our results about ω-algebra carry over to this part of duration calculus. Of course, the example of a gas burner is a very small example for reactive systems. But it points out how to use algebraic structure for DC and shows the advantages of such an algebraic semantics.

6 Conclusion and Outlook

We have given a connection between the Duration Calculus developed by Zhou et al. and the theory of semirings (quantales). This allows us to calculate very easily with safety requirements and to only consider the few axioms given by semirings and Kleene algebras, respectively. Furthermore, we can re-use all results from Kleene algebra, ω-algebra and so on for DC. In [7] another interpretation of reactive systems is given, which uses a trajectory-based model instead of an interval-based one. With this model, we can also specify safety criteria and other requirements.

The aim of further work is to develop an extension of DC using left semirings [9]. This will give the possibility to simulate not only finite processes, but even infinite ones. Regrettably, the calculations using (right) detachments cannot easily be adapted for left semirings. Thus we have to modify the calculations to get similar results.

As another aim, we will have a look on the ITL-extending logics, like the propositional calculus of Venema [11] or the Neighbourhood Logic of Zhou and Hansen [15]. A first short study suggests the conjecture that these logics form also Boolean quantales. Venema introduces in [11] the two modalities T and D, which are expanding in the sense that the truth value of formulas $\phi \mathrm{T} \psi$ and $\phi \mathrm{D} \psi$ on an interval depends on intervals 'outside'. So $\phi \mathrm{T} \psi$ holds on an interval $[b, e]$ iff there exist an interval $[e, c], c \geq e$ in the future such that ϕ holds on $[e, c]$ and ψ holds on $[b, e]$. Symmetrically, $\phi \mathrm{D} \psi$ describes the situation for an 'outside' interval in the past. It seems that we get the following relationship:

$$\begin{aligned} \phi \mathrm{T} \psi \text{ holds on } [b, e] &\Leftrightarrow [b, e] \leq |[\![\phi]\!]\rangle [\![\psi]\!] \\ \phi \mathrm{D} \psi \text{ holds on } [b, e] &\Leftrightarrow [b, e] \leq \langle [\![\phi]\!]| [\![\psi]\!] \end{aligned}$$

where $[\![\psi]\!]$ is the set of all intervals where ψ holds.

The Neighbourhood Logic expands DC, too, and introduces *left* and *right neighbourhoods* as primitive interval modalities. Neighbourhood Logic supports formal specification and verification of liveness and fairness by introducing the operators $\diamond_l$ and $\diamond_r$. With $\diamond_l \phi$ ($\diamond_r \phi$) one can reach the left (right) neighbourhoods of the beginning (ending) point of an interval. Probably, $\diamond_l \phi$ and $\diamond_r \phi$ can be expressed by

$$\begin{aligned} \diamond_l \phi \text{ holds on } [b, e] &\Leftrightarrow [b, e] \leq |[\![\phi]\!]\rangle \top \\ \diamond_r \phi \text{ holds on } [b, e] &\Leftrightarrow [b, e] \leq \langle [\![\phi]\!]| \top \end{aligned}$$

If these laws hold, we can embed the Neighbourhood Logic of Zhou and Hansen as well as the propositional calculus of Venema in the more general concept of Boolean quantales so that our results would carry over this framework.

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