Backwards and Forwards in Separation Algebra

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(joint work with C. Bannister and G. Klein)

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Separation Logic (Reynolds, O'Hearn et al.)

- Extension to Hoare Logic
- Based on Separation *Algebras* of abstract heaps
- Captures the notion of *disjointness* in the world

Separation Logic (Reynolds, O'Hearn et al.)

Motivation

- Pointer programs are hard to reason about

$$\{p \mapsto a\}$$
$$\text{delete } p$$
$$\{p \not\mapsto \_\}$$

The Frame Problem

Separation Logic (Reynolds, O'Hearn et al.)

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$$\{p \mapsto a \wedge p' \mapsto b\}$$
$$\text{delete } p$$
$$\{p \not\mapsto \_ \wedge p' \mapsto b\}$$

The Frame Problem

Separation Logic (Reynolds, O'Hearn et al.)

Motivation

- Pointer programs are hard to reason about

$$\{p \mapsto a \wedge p' \mapsto b \wedge p \neq p'\}$$
$$\text{delete } p$$
$$\{p \not\mapsto \_ \wedge p' \mapsto b \wedge p \neq p'\}$$

The Frame Problem

Separation Logic (Reynolds, O'Hearn et al.)

Motivation

- $s, h \models P$
  where *s* is a store, *h* is a heap, and *P* is an *assertion* over the given store and heap

$$
\begin{array}{ll}
& s, h \models P * Q \\
\Leftrightarrow & \exists h_1, h_2.\ h_1 \perp h_2 \text{ and} \\
& h = h_1 \cup h_2 \text{ and } s, h_1 \models P \text{ and } s, h_2 \models Q
\end{array}
$$

Separation Logic (Reynolds, O'Hearn et al.)

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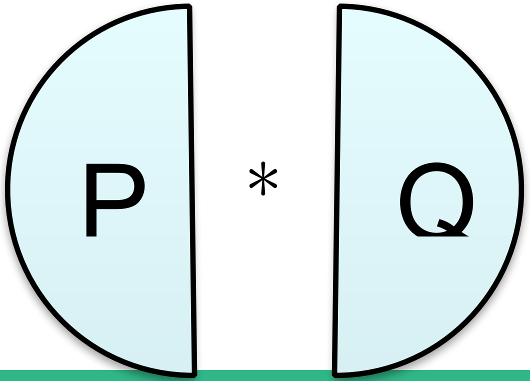


Frame Rule

$$\frac{\{P\}\ C\ \{Q\}}{\{P * R\}\ C\ \{Q * R\}}\quad (mod(C) \cap fv(R) = \emptyset)$$

- R is the 'Frame'
  - Extending an environment with a disjoint portion changes nothing
  - Local Reasoning
  - Compositional

Separation Algebras

- Separation logic can be lifted to algebra

- Allows abstract reasoning
- Transfers knowledge
- Ideal for interactive and automated theorem proving

Separation Algebras (Calcagno et al.)

- partial commutative monoid
  partial plus (+), and neutral element (0)

- h # h' captures the 'definedness' or partiality of (+)

- 0 is the empty heap

$$x + 0 = x \qquad x \# 0$$

$$s, h \models P * Q \;\Leftrightarrow\; \exists h_1, h_2.\; h_1 \# h_2 \wedge h = h_1 + h_2 \wedge P(h_1) \wedge Q(h_2)$$

Algebra of Assertions (Dang et al.)

- Set-based semantics

$$[\![p]\!] \Leftrightarrow \{(s,h) : s,h \models p\} \ .$$

$$\begin{array}{rcl} [\![p * q]\!] & = & [\![p]\!] \dot{\cup} [\![q]\!] \\ P \dot{\cup} Q & \Leftrightarrow & \{(s, h \cup h') : (s,h) \in P \wedge (s,h') \in Q \\ & & \wedge\ doms(h) \cap dom(h') = \emptyset\} \ . \end{array}$$

Separating Implication

Magic Wand

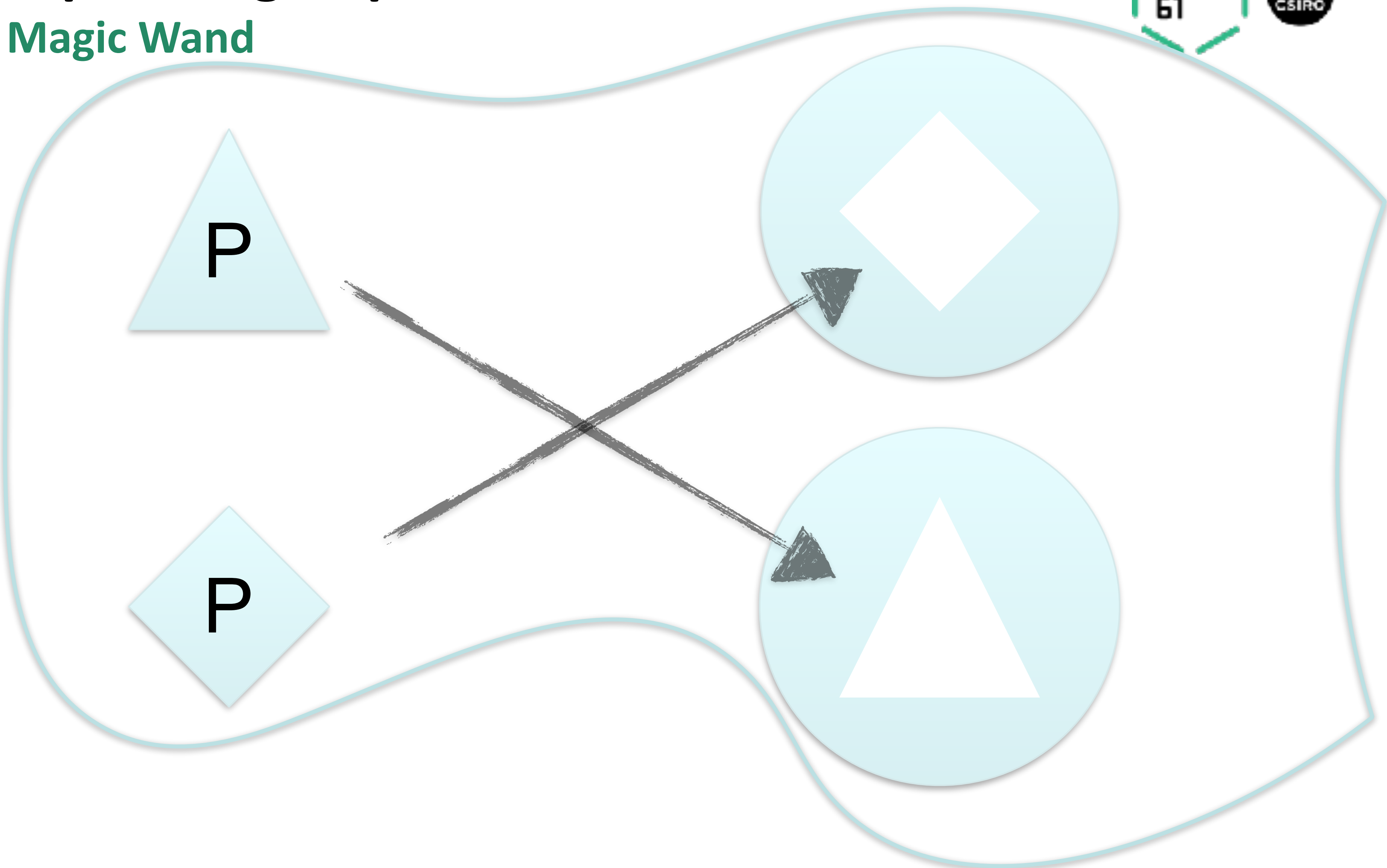
- Separating Implication $P \mathrel{-\!\!*} Q$
  - Extending by P produces Q over the combination

- Describes a *mapping* between heaps and 'holes'

$$s, h \models P \mathrel{-\!\!*} Q \quad \Leftrightarrow \quad \forall h'.\ (h' \perp h \text{ and } s, h' \models P) \\ \text{implies } s, h' \cup h \models Q$$

Separating Implication

Magic Wand



Separating Implication

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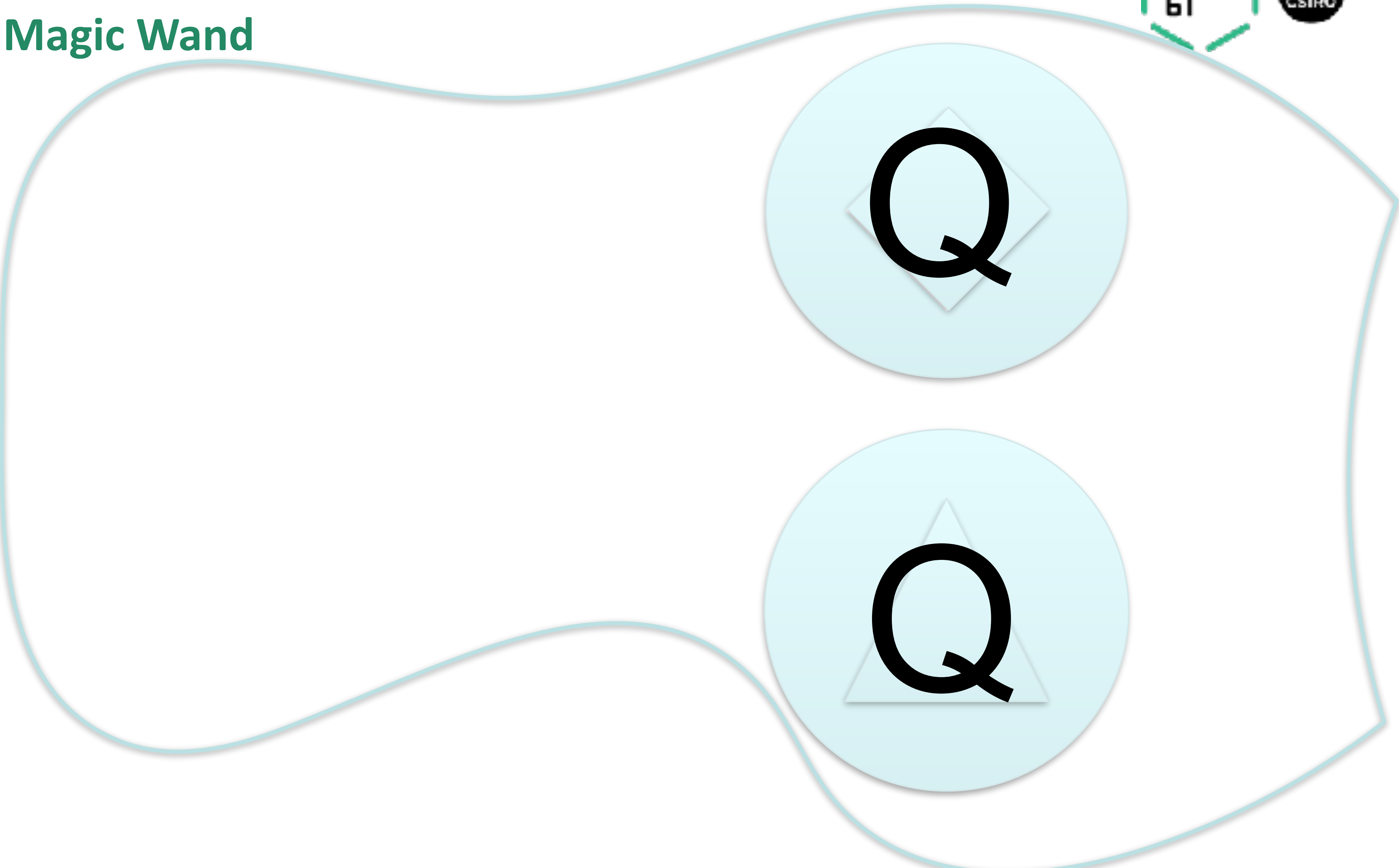

Conjunction version Implication

- Podus ponens

$$\frac{s, h \models Q * (Q -\!\!* P)}{s, h \models P}$$

Conjunction version Implication

- Podus ponens

$$\llbracket Q * (Q \mathrel{-\!\!*} P) \rrbracket \;\subseteq\; \llbracket P \rrbracket$$

Conjunction version Implication

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$$Q * (Q \mathrel{-\!\!*} P) \;\Rightarrow\; P$$

Conjunction version Implication

- Podus ponens

$$Q * (Q \mathrel{-\!\!*} P) \;\Rightarrow\; P$$

- Currying/decurrying

$$(P * Q \;\Rightarrow\; R) \;\Leftrightarrow\; (P \;\Rightarrow\; Q \mathrel{-\!\!*} R)$$

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$$(P * Q \;\Rightarrow\; R) \;\Leftrightarrow\; (P \;\Rightarrow\; Q \mathrel{-\!\!*} R)$$

Galois connection

Relationships between Operators

$$\boxed{P * Q} \xleftrightarrow{\text{Galois}} \boxed{P \mathrel{-\!\!*} Q}$$

Backwards Reasoning

- Backward reasoning / reasoning in weakest precondition style
- for given postcondition Q and given program C, determine weakest precondition $wp(C, Q)$ such that

$$\{wp(C, Q)\}\ C\ \{Q\}$$

  is valid Hoare triple

- but what about separation logic where frames occur?

$$\{P * R\}\ C\ \{Q * R\}$$

(problem with frame calculation)

Example

- Program:

$$\texttt{copy\_ptr}\ p\ p' \quad = \quad \texttt{do}\{x \leftarrow \texttt{get\_ptr}\ p;\ \texttt{set\_ptr}\ p'\ x\}$$

- Specification (Hoare triple)

$$\{\!|p \mapsto x * p' \mapsto \_ * R|\!\}\ \texttt{copy\_ptr}\ p\ p'\ \{\!|p \mapsto x * p' \mapsto x * R|\!\}$$

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$$\{\!|p \mapsto x * p' \mapsto \_ * R|\!\}\ \texttt{copy\_ptr}\ p\ p'\ \{\!|p \mapsto x * p' \mapsto x * R|\!\}$$

- Assume the program occurs in larger context and the postcondition is

$$\{\!|R'' * p' \mapsto v * a \mapsto \_ * p \mapsto v * R'|\!\}$$

Backwards Reasoning II

- from Galois connection we get

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \quad \Leftrightarrow \quad (\forall R.\ \{P * (Q \mathrel{-\!\!*} R)\}\ C\ \{R\})$$

- used to transform specifications
  for example

$$\{p \mapsto \_ * R\}\ \mathrm{set\_ptr}\ p\ v\ \{p \mapsto v * R\}$$

Backwards Reasoning II

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$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \quad \Leftrightarrow \quad (\forall R.\ \{P * (Q \mathrel{-\!\!*} R)\}\ C\ \{R\})$$

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$$\{p \mapsto \_ * (p \mapsto v \mathrel{-\!\!*} R)\}\ \mathrm{set\_ptr}\ p\ v\ \{R\}$$

Backwards Reasoning III

- Allows now full backwards reasoning without calculating the frame in every step
- Supported in Isabelle/HOL
- Easy patterns (alternation between implication and conjunction) allow automated simplifications

Forward Reasoning

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \quad \Leftrightarrow \quad (\forall R.\ \{R\}\ C\ \{??\})$$

Forward Reasoning II

- Ideal world

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \quad \Leftrightarrow \quad (\forall R.\ \{R\}\ C\ \{Q * (P \multimap\!\circledast\ R)\})$$

More Separation Logic

- there is another operator in the literature: septraction

$$s, h \models P \;-\!\circledast\; Q$$

$$\Leftrightarrow\; \exists h_2.\, h \text{ subheap of } h_2 \text{ and } s, h_2 - h \models P \text{ and } s, h_2 \models Q$$

- algebraically:

$$P \;-\!\circledast\; Q \;\Leftrightarrow\; \neg(P \mathrel{-\!\!*} (\neg Q))$$

Forward Reasoning II

- Ideal world seemingly impossible

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \quad \Leftrightarrow \quad (\forall R.\ \{R\}\ C\ \{Q * (P \multimap\!\!\circledast\ R)\})$$

- Cannot describe what happens in cases where precondition does not hold

$$\{emp\}\ \text{delete}\ p\ \{??\}$$

Forward Reasoning II

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Relationships between Operators

$$\begin{tikzcd} P * Q \arrow[rr, leftrightarrow, bend left, "\text{Galois}"] & & P \mathrel{-\!\!*} Q \arrow[d, leftrightarrow, "\text{dual}"'] \\ & & P \mathrel{-\!\!\circledast} Q \end{tikzcd}$$

Separating 'Coimplication'

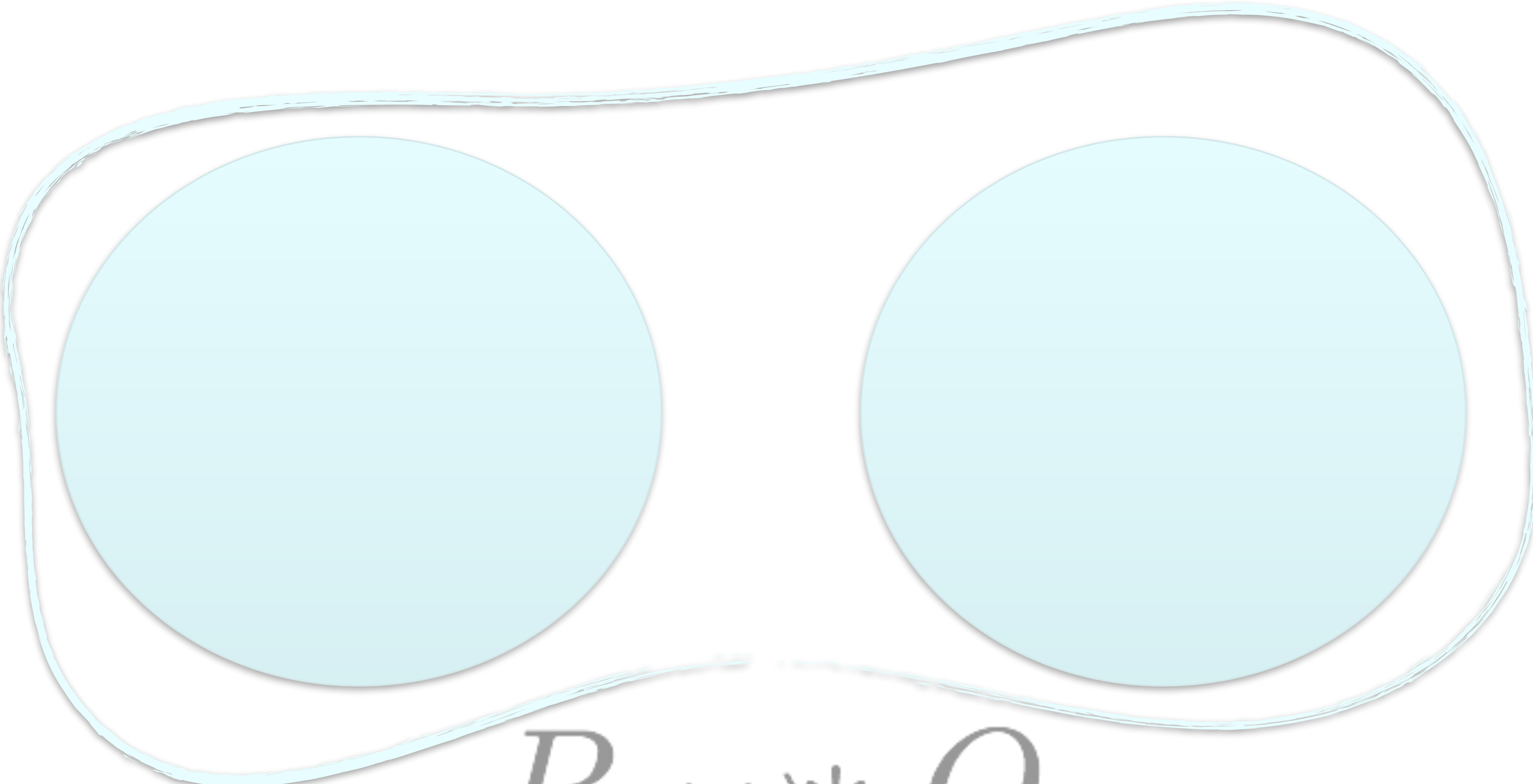
Magic Snake

- $$P \rightsquigarrow\!\!\ast\, Q \;\Leftrightarrow\; \neg(P * (\neg Q))$$

- Removing P produces Q over the reduction

- Every time we can find a P in our heap, the rest of the heap is a Q

Separating Coimplication



$$P \rightsquigarrow\!\!* \; Q$$

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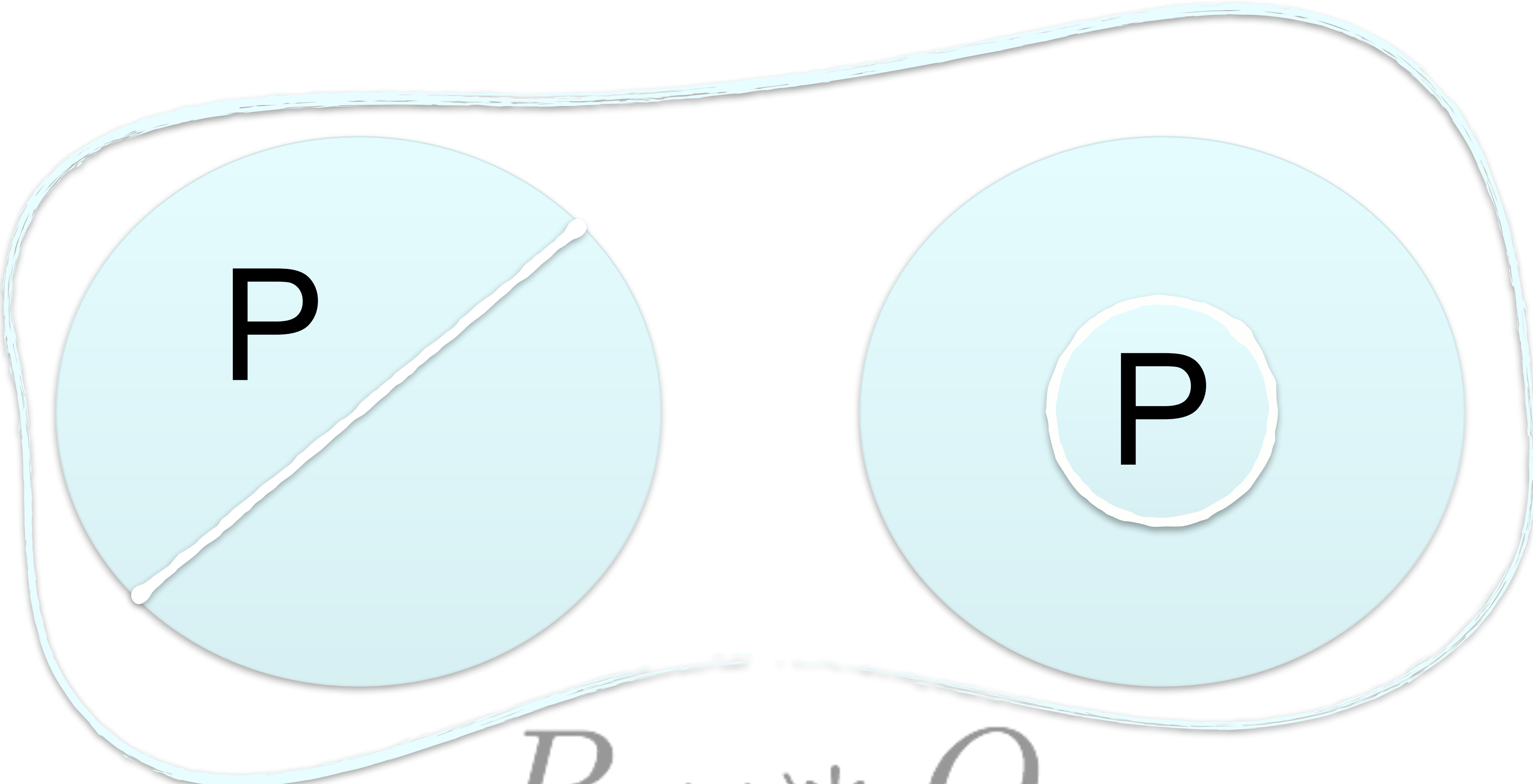

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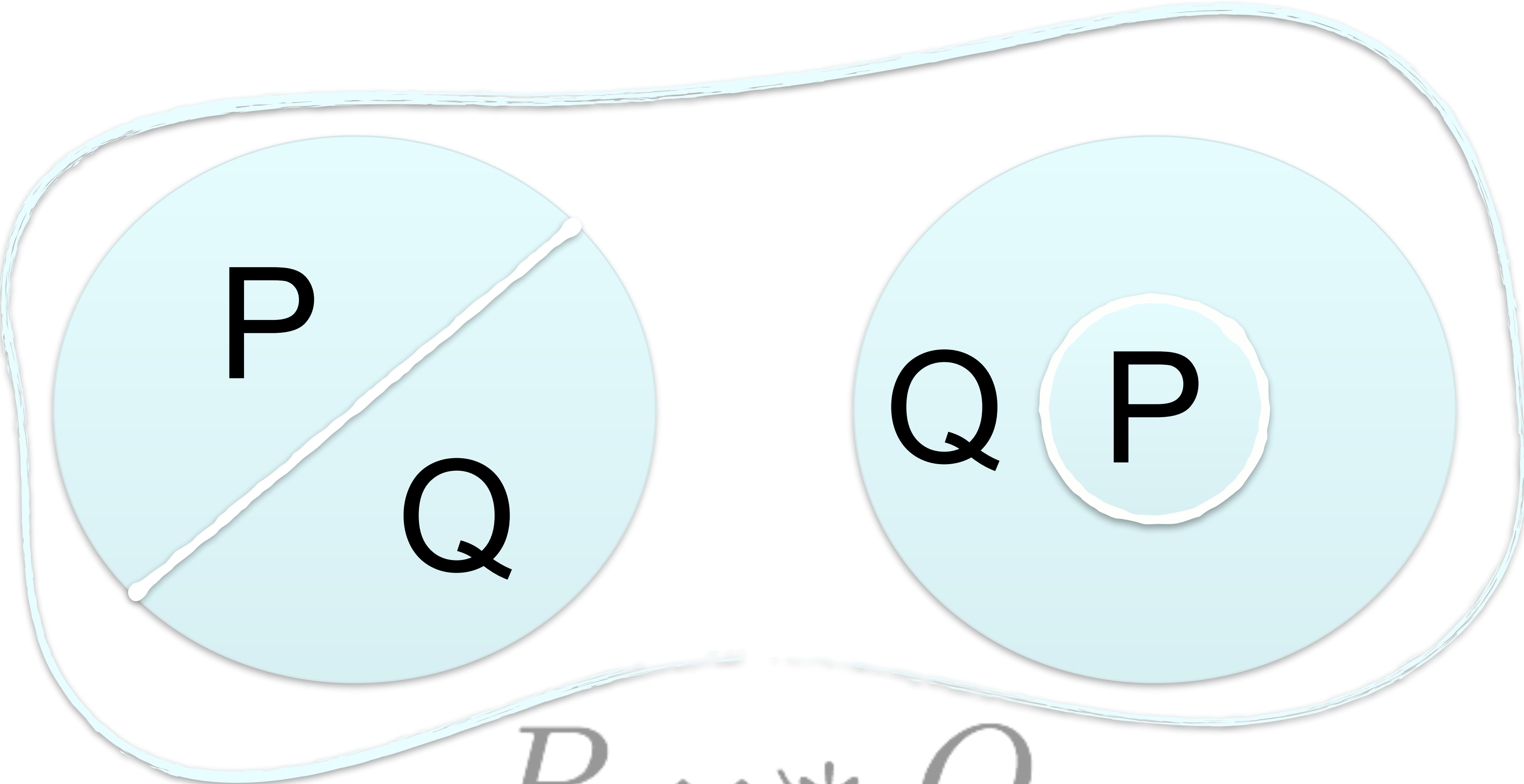

$$P \rightsquigarrow\!\!* \; Q$$

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Separating 'Coimplication'

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- $$P \leadsto\!\!\ast\, Q \;\Leftrightarrow\; \neg(P * (\neg Q))$$

$$(P \multimap\!\circledast\; Q \;\Rightarrow\; R) \;\Leftrightarrow\; (Q \;\Rightarrow\; (P \leadsto\!\!\ast\, Q))$$

(Galois connection)

- many properties come for free from the Galois connection

Relationships between Operators
(complete)

$$
\begin{tikzcd}
\boxed{P * Q} \arrow[rr, leftrightarrow, bend left, "\text{Galois}"] \arrow[d, leftrightarrow, "\text{dual}"] & & \boxed{P \mathrel{-\!\!*} Q} \arrow[d, leftrightarrow, "\text{dual}"'] \\
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$$

Specifications with Separating Coimplication

- P not satisfied by any subhead

$$P \leadsto\!\!\!* \; false$$

- specification of delete

$$\{p \mapsto \_ \leadsto\!\!\!* \; R\} \text{ delete } p \; \{R\}$$

Back to Forward Reasoning

- Ideal world seemingly impossible

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \ \not\Leftrightarrow\ (\forall R.\ \{R\}\ C\ \{Q * (P \multimap\!\circledast\ R)\})$$

- Relax specifications/requirements

$$\{P * R\}\ C\ \{Q * R\}$$

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$$\{P \mathbin{\rightsquigarrow\!\!*} R\}\ C\ \{Q * R\}$$

- another example

$$\{p \mapsto \_ \mathbin{\rightsquigarrow\!\!*} R\}\ \mathrm{set\_ptr}\ p\ v\ \{p \mapsto v * R\}$$

Forward Reasoning III

- Ideal world seemingly impossible

$$(\forall R.\ \{P * R\}\ C\ \{Q * R\}) \;\not\Leftrightarrow\; (\forall R.\ \{R\}\ C\ \{Q * (P \multimap\!\circledast\ R)\})$$

- By Galois connections and dualities we get a rule for forward reasoning

$$(\forall R.\ \{P \rightsquigarrow\!\!* R\}\ C\ \{Q * R\}) \;\Leftrightarrow\; (\forall R.\ \{R\}\ C\ \{Q * (P \multimap\!\circledast\ R)\})$$

Forwards Reasoning IV

- allows backwards reasoning without calculating the frame in every step
- supported in Isabelle/HOL
- easy patterns (alternation between implication and conjunction) allow automated simplifications

Forward Reasoning (Problems)

- we restricted ourselves to partial correctness
  - no problem for backwards reasoning
  - but for forward reasoning postcondition does not need to exist

- rules are only valid because we deal with partial correctness

$$\{P\}\ C\ \{Q\} \ \Leftrightarrow\ \forall s.\ P(s) \rightarrow (\forall s'.\ Some\ s' = (C\ s) \rightarrow Q(s'))$$

- if failure occurs anything is possible

$$\{p \not\mapsto \_\}\ \mathrm{set\_ptr}\ p\ v\ \{\mathrm{P{=}NP}\}$$

Unified Correctness

- introduce explicit failure state
- always describe what actually occurs

$$\{P\}\ C\ \{Q\} \;\Leftrightarrow\; \forall s.\ P(s) \to Q(C(s))$$

- requirements:
  - failed program execution stays failed

$$\{\text{fail}\}\ C\ \{\text{fail}\}$$

  - failure is separate from False
  - we can determine whether or not we succeeded

  - closely related to general correctness by Jacobs & Gries (1985)

Extending the Model

- New Heap Model

  - Same as standard heap model, but we add a boolean flag for failure

$$[p \mapsto v, q \mapsto v' ..] \to ([p \mapsto v, q \mapsto v' ..], True)$$

$$(h, False) + (h', -) = (h + h', False)$$

- “Infinitely” many failure states

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- “Infinitely” many failure states

- **But: Galois Connections do not hold any longer!**

Extending the Model (New Ops)

- New Septraction operator for grabbing resources

Old

$$s, h \models P \;-\!\!\circledast\; Q$$

$$\Leftrightarrow\; \exists h_2.\, h \text{ subheap of } h_2 \text{ and } s, h_2 - h \models P \text{ and } s, h_2 \models Q$$

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$$\Leftrightarrow\;\; \exists h_1, h_2.\; \models P \text{ and } s, h_1 \models Q \text{ and}$$

$$h \perp h_1,\; h_2 = h + h_1$$

Extending the Model (New Ops)

- New Septraction operator for grabbing resources

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$$s, h \models P \multimap\!\circledast\; Q$$
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New

$$s, h \models P \multimap\!\circledast\; Q$$
$$\Leftrightarrow\; \exists h_1, h_2.\; \models P \text{ and } s, h_1 \models Q \text{ and}$$
$$\textbf{if } \mathrm{flag}(h) \textbf{ then } h \perp h_1,\; h_2 = h + h_1$$
$$\textbf{else } \mathrm{flag}(h_2) \rightarrow (\mathrm{flag}(h_1) \rightarrow h_1 \perp h_2) \wedge (\mathrm{flag}(h) \rightarrow h \perp h_2)$$

Extending the Model (New Ops II)

- Desired properties are satisfied

- **Consuming**: If the resource is there, we succeed

$$s, h \models (p \mapsto v) \multimap\!\circledast (p \mapsto v) \Rightarrow h = (\mathrm{emp}, true)$$

- **Collapsing:** Once crashed, remain crashed

$$s, h \models P \multimap\!\circledast \text{`fail'} \Rightarrow h = (\_, false)$$

- **Paraconsistent:** Removing something that didn't exist yield failure

$$s, h \models p \mapsto \_ \multimap\!\circledast \mathrm{emp} \Rightarrow h = (\_, false)$$

The Good and The Bad

- New operators satisfy Galois connections and dualities

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- Separation algebra is identical to the ‘old’ in case of **no failure**
- In case of failure, associativity of separating conjunction is lost

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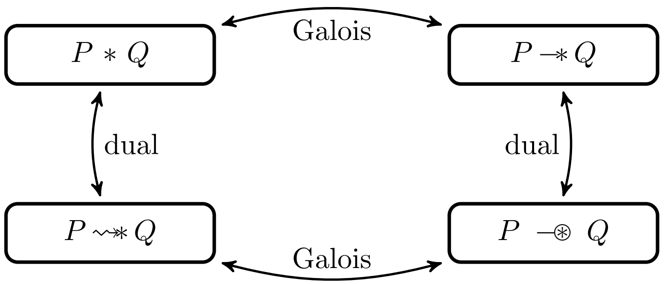

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- Alternative idea (R. Gore): use different negation (intuitionistic logic or Sheffer stroke)

Conclusion

- Framework for
  backwards reasoning using weakest preconditions and
  forward reasoning using strongest postconditions for Partial and Unified Correctness

- Automation

- Basic examples demonstrated
  - e.g. Linked-List Reverse
  - for forward reasoning: big case study: system init on seL4

Thank you

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