Verification of Relational Programs and Approximation Algorithms

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CSIRO's Digital Productivity business unit and NICTA have joined forces to create digital powerhouse Data61

Motivation

towards more automation in program verification

- functional correctness
- use algebra to improve proof automation
- use pre-/postconditions (Hoare-style reasoning)

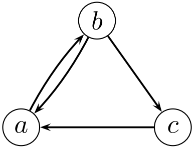
at the moment

- look at ‘simple’ and well-known while programs (pre-/postconditions, invariant proofs)
- use relational algebra
- limited to algorithms where data structure can be modelled by (relational) algebra
- investigate the power of cardinalities over relational algebra

Relation Algebra: The Standard Model

the standard model are relations (sets over $M \times M$)

$$\{(a,b),(b,a),(b,c),(c,a)\}$$



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$n \times n$ matrices

operations/constans:

- $\cup, \cap, ^{-}$ are set theoretic definitions
- $R;S = \{(a,c) \mid \exists b : (a,b) \in R \wedge (b,c) \in S\}$
- $R^{\mathsf{T}} = \{(b,a) \mid (a,b) \in R\}$
- $\mathsf{O} = \emptyset$, $\mathsf{L} = M \times M$, $\mathsf{I} = \{(a,a) \mid a \in M\}$

Relation Algebra

a *relation algebra* is a structure $(A, \cup, ;, \overline{}, ^{\mathsf{T}}, \mathsf{I})$ such that

- $(A, \cup, \overline{})$ is Boolean algebra, i.e.,
  $(Q \cup R) \cup S = Q \cup (R \cup S), \quad Q \cup R = R \cup Q,$
  $R = \overline{\overline{R} \cup \overline{S}} \cup \overline{\overline{R} \cup S}$
- provides an operation for *composition*
  $(Q;R);S = Q;(R;S), \quad (Q \cup R);S = Q;S \cup R;S \quad R;\mathsf{I} = R$
- defines an operation of *conversion*
  $R^{\mathsf{T}^{\mathsf{T}}} = R, \quad (R \cup S)^{\mathsf{T}} = R^{\mathsf{T}} \cup S^{\mathsf{T}}, \quad R^{\mathsf{T}};\overline{R;S} \cup \overline{S} = \overline{S}$

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additional constants/operations:

- intersection: $R \cap S = \overline{\overline{R} \cup \overline{S}}$
- order: $R \subseteq S \Leftrightarrow R \cup S = S$
- $R^* = \bigcup_{i \geq 0} R^i = I \cup R \cup R^2 \cup \ldots$
  (first-order characterisation possible)
- smallest and greatest element: $\mathsf{O} = R \cap \overline{R}$, $\mathsf{L} = R \cup \overline{R}$

Warming Up: Reflexive-Transitive Closure

input R

$C, v := \mathsf{I}, \mathsf{O}$

while $v \neq R;\mathsf{L}$ **do**

  let $p = \mathit{point}(R;\mathsf{L} \cap \overline{v})$;

  $C, v := C \cup C;p;p^{\mathsf{T}};R;C,\ \ v \cup p$;

od

return C

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$\{C = R^*\}$

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$\{True\}$

$C, v := \mathsf{I}, \mathsf{O}$

$\{C = (R \cap v)^* \ \wedge \ v = v;\mathsf{L}\}$

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- $Inv_0(R, C, v) \Leftrightarrow C = (R \cap v)^*$
  $Inv_1(v) \Leftrightarrow v = v;\mathsf{L}$

Warming Up: Reflexive-Transitive Closure

- correctness proof: simple exercise ?
- $Inv_0(R, C, v) \Leftrightarrow C = (R \cap v)^*$
  $Inv_1(v) \Leftrightarrow v = v;\mathsf{L}$
- p is point $\Leftrightarrow p;\mathsf{L} = p \wedge \mathsf{L};p = \mathsf{L} \wedge p;p \subseteq \mathsf{I}$

Warming Up: Reflexive-Transitive Closure

- correctness proof: simple exercise ?
- $Inv_0(R, C, v) \Leftrightarrow C = (R \cap v)^*$
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- p is point $\Leftrightarrow p;\mathsf{L} = p \wedge \mathsf{L};p = \mathsf{L} \wedge p;p \subseteq \mathsf{I}$
- proof automation
  (e.g. the automated theorem prover Prover9)

Establishment	
$Inv_0(R, \mathsf{I}, \mathsf{O}) \wedge Inv_1(\mathsf{O})$	$0\,s$
Post-Condition	
$v = R;\mathsf{L} \wedge Inv_0(R, C, v) \wedge Inv_1(v) \Rightarrow C = R^*$	$0\,s$
Maintenance	
$Inv_1(v) \wedge p \text{ is point} \wedge p \subseteq R;\mathsf{L} \cap \overline{v} \;\Rightarrow\; Inv_1(v \cup p)$	$1\,s$
$Inv_0(R, C, v) \wedge p \text{ is point} \wedge p \subseteq R;\mathsf{L} \cap \overline{v}$ $\Rightarrow\; Inv_0(R, C \cup C;p;p^{\mathsf{T}};R;C, v \cup p)$	–

Warming Up: Reflexive-Transitive Closure

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Maintenance	
$Inv_1(v) \wedge p$ is point $\wedge\ p \subseteq R;\mathsf{L} \cap \overline{v} \;\Rightarrow\; Inv_1(v \cup p)$	$1\,s$
$Inv_0(R, C, v) \wedge p$ is point $\wedge\ p \subseteq R;\mathsf{L} \cap \overline{v}$ $\Rightarrow\; Inv_0(R, C \cup C;p;p^{\mathsf{T}};R;C, v \cup p)$	$0\,s$

add 3 theorems about the operation $*$

Conclusion

- shows partial correctness only
  total correctness has to be shown separately
- often automated reasoning helps – but not always
- perfect candidate for interactive theorem proving/proof assistants (preferable with some bits of proof automation)

- more algorithms verified
  - topological sorting
  - node colouring
  - matching algorithms
  - ...
- verification of Relational-While Programs can be done (in)equationally and automatically;
  in particular RA seems to be well suited for graph problems

Verification of Relational-While Program

- can be done equationally and automatically

Verification of Relational-While Program

- can be done equationally and automatically
- BUT: what about cardinalities

Cardinalities on Relation Algebra

- introduced by Prof. Kawahara [Kaw06]
- operation $|.|$ over relation algebra
- Prof. Kawahara looked at basic graph theory, such as the theorem of Hall and König
- we used his approach for verification

Cardinalities on Relation Algebra

(C1) if R is finite, then $|R| \in \mathbb{N}$, and
$|R| = 0$ iff $R = \mathsf{O}$

(C2) $|R| = |R^{\mathsf{T}}|$

(C3) if R and S are finite, then
$|R \cup S| = |R| + |S| - |R \cap S|$.

(C4) if Q is univalent ($Q^{\mathsf{T}};Q \subseteq \mathsf{I}$), then
$|R \cap Q^{\mathsf{T}};S| \leq |Q;R \cap S|$, and
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(C5) $|\mathsf{I} \quad| =$

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(C5) $|\mathsf{I}_{11}| = 1$

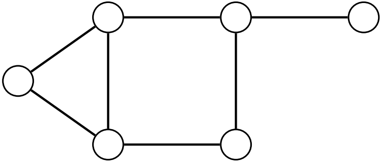
we have to calculate in a heterogenous setting ($m \times n$ matrices)

Relations, Points, Vectors and Cardinalities

useful properties (sanity check)

- $|.|$ is monotone, i.e. $R \subseteq S \Rightarrow |R| \subseteq |S|$
- if p is point, then $|p| = 1$
- if v is vector, then $|v| == |\bigcup_{p \in \mathcal{P}_v} p| = \sum_{p \in \mathcal{P}_v} |p|$
- if R is univalent and S is a mapping, then $|R;S| = |R|$
- if R is symmetric, P is injective and Q is univalent, then $|R \cap P;Q^\mathsf{T}| = |R;P \cap Q|$

Minimum Vertex Covers



- problem is NP-complete
- approximation algorithm of Garvil and Yannakakis
- cardinalities are used to give quality of approximation

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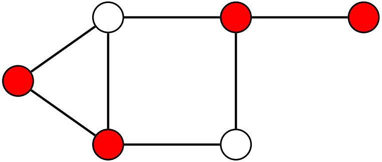

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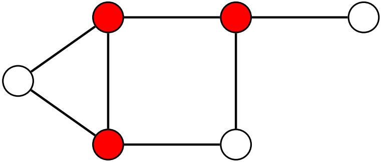

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Minimum Vertex Covers

input R

$c, S \quad := \mathsf{O}_{X\mathbf{1}}, R \quad ;$

while $S \neq \mathsf{O}$ **do**

  let $e \; = edge(S)$;

  $c, S \qquad := \; c \cup e;\mathsf{L}, \; S \cap \overline{e;\mathsf{L} \cup \mathsf{L};e} \qquad ;$

od

return c

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$\{R \subseteq c;\mathsf{L} \cup (c;\mathsf{L})^{\mathsf{T}},$
$\ \forall d : X \leftrightarrow \mathbf{1} \bullet R \subseteq d;\mathsf{L} \cup (d;\mathsf{L})^{\mathsf{T}} \Rightarrow |c| \leq 2 \cdot |d|\}$

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$\{M \subseteq R,\ M = M^{\mathsf{T}},\ M;M \subseteq \mathsf{I},\ M;\mathsf{L} \cap S = \mathsf{O}$ $\quad M$ aux. variable
$R \cap \overline{S} \subseteq c;\mathsf{L} \cup (c;\mathsf{L})^{\mathsf{T}},\ S \subseteq R,\ S = S^{\mathsf{T}},\ |c| \leq |M|\}$

while $S \neq \mathsf{O}$ **do**

let $e = edge(S);$

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Correctness Proof

- most invariant proofs are equational and "easy";
  they are verified using the proof assistant Coq
  (there are just more and invariants)
- short proofs
- proof automation would be useful
  (e.g. try Isabelle's tool sledgehammer)

Correctness Proof: Cardinalities

Invariant

$$
\begin{aligned}
|c \cup e;\mathsf{L}| &\leq |c| + |e;\mathsf{L}| - |c \cap e;\mathsf{L}| && \text{// by (C3)} \\
&= |c| + |e;\mathsf{L}| && \text{// isotonicity} \\
&\leq |M| + |e;\mathsf{L}| && \text{// invariant} \\
&= |M| + |e| && \text{// } e \text{ vector} \\
&= |M| + |e| - |M \cap e| && \text{// as } M \cap e = \mathsf{O}\text{, by (C1)} \\
&= |M \cup e| && \text{// by (C3)}
\end{aligned}
$$

Correctness Proof: Cardinalities

Postcondition

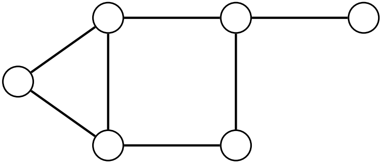
$$
\begin{aligned}
|M| &= |M;(d \cup \overline{d})| && \text{// } M \text{ univalent, } d \cup \overline{d} \text{ mapping, aux. Lemma} \\
&= |M;d \cup M;\overline{d}| \\
&\leq |M;d| + |M;\overline{d}| && \text{// by (C3), isotony} \\
&\leq |M;d| + |R;\overline{d}| && \text{// invariant, isotonicity} \\
&\leq |M;d| + |d| && \text{// as } R;\overline{d} \subseteq d\text{, isotonicity} \\
&= |\mathsf{L} \cap M^{\mathsf{T}\mathsf{T}};d| + |d| \\
&\leq |M^{\mathsf{T}};\mathsf{L} \cap d| + |d| && \text{// } M^{\mathsf{T}} \text{ univalent, by (C4)} \\
&\leq |d| + |d| && \text{// isotonicity} \\
&= 2 \cdot |d|
\end{aligned}
$$

Adaptation to Hitting Sets

- same algorithm, different relation algebra (calculating on incidence relation $I : X \leftrightarrow E$)
- this models hypergraphs (edges are set of nodes)
- cardinality of all maximal hyperedges: $\max\{|I;p| \mid p : E \leftrightarrow \mathbf{1} \text{ point}\}$
- algorithm generalises to hyper graph with approximation

$$\forall d : X \leftrightarrow \mathbf{1} \bullet \mathsf{L} = I^{\mathsf{T}};d \Rightarrow |c| \leq k \cdot |d|$$

Maximum Cuts (Max-Cut)



- problem is NP-complete
- approximation algorithm
- cardinalities are used to give quality of approximation
- cardinality of cut: $|R \cap (s;\overline{s}^{\mathsf{T}} \cup \overline{s};s^{\mathsf{T}})|$

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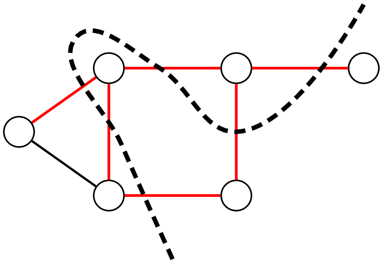

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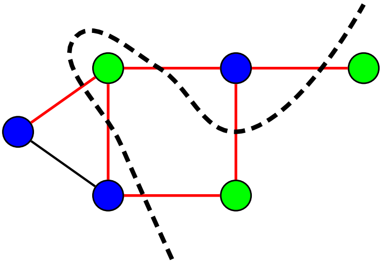

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Minimum Vertex Covers

input R

$v, s, t := \mathsf{L}_{X\mathbf{1}}, \mathsf{O}, \mathsf{O};$

while $v \neq \mathsf{O}$ **do**

    let $p \ = point(v);$

    if $|R;p \cap s| < |R;p \cap t|$

        then $v, s := v \cap \overline{p}, s \cup p$

        else $v, t := v \cap \overline{p}, t \cup p$

    fi

od

return s

Minimum Vertex Covers

input R

$\{R \subseteq \bar{\mathsf{I}},\ R = R^{\mathsf{T}}\}$

$v, s, t := \mathsf{L}_{X\mathbf{1}}, \mathsf{O}, \mathsf{O};$

$\{s \cap t = \mathsf{O},\ s \cup t = \overline{v},\ |R \cap (s;s^{\mathsf{T}} \cup t;t^{\mathsf{T}})| \leq |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})|\}$

while $v \neq \mathsf{O}$ **do**

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    else $v, t := v \cap \overline{p}, t \cup p$

  fi

od

return s

$\{\forall c : X \leftrightarrow \mathbf{1} \bullet |R \cap (c;\overline{c}^{\mathsf{T}} \cup \overline{c};c^{\mathsf{T}})| \leq 2 \cdot |R \cap (s;\overline{s}^{\mathsf{T}} \cup \overline{s};s^{\mathsf{T}})|\}$

Correctness Proof

- non-cardinality proofs are again standard;
  they are verified using the proof assistant Coq
- approximation bound is $\frac{1}{2}$:
  $\forall c : X \leftrightarrow \mathbf{1} \bullet |R \cap (c;\overline{c}^{\mathsf{T}} \cup \overline{c};c^{\mathsf{T}})| \leq 2 \cdot |R \cap (s;\overline{s}^{\mathsf{T}} \cup \overline{s};s^{\mathsf{T}})|$

Correctness Proof: Cardinalities

Invariant

$$
\begin{array}{ll}
|R \cap ((s \cup p);(s \cup p)^\top \cup t;t^\top)| & \\
= |(R \cap (s;s^\top \cup t;t^\top \cup s;p^\top \cup p;s^\top \cup p;p^\top)| & \\
= |(R \cap (s;s^\top \cup t;t^\top)) \cup (R \cap s;p^\top) \cup (R \cap p;s^\top)| & \text{// 1st aux. result} \\
\leq |R \cap (s;s^\top \cup t;t^\top)| + |R \cap s;p^\top| + |R \cap p;s^\top| & \text{// by (C3)} \\
\leq |R \cap (s;t^\top \cup t;s^\top)| + |R \cap s;p^\top| + |R \cap p;s^\top| & \text{// invariant} \\
= |R \cap (s;t^\top \cup t;s^\top)| + |R \cap p;s^\top| + |R \cap p;s^\top| & \text{// by (C2), } R = R^\top \\
= |R \cap (s;t^\top \cup t;s^\top)| + |R;p \cap s| + |R;p \cap s| & \text{// 1st aux. lemma} \\
< |R \cap (s;t^\top \cup t;s^\top)| + |R;p \cap t| + |R;p \cap t| & \text{// as } |R;p \cap s| < |R;p \cap t| \\
= |R \cap (s;t^\top \cup t;s^\top)| + |R \cap p;t^\top| + |R \cap p;t^\top| & \text{// aux. lemma} \\
= |R \cap (s;t^\top \cup t;s^\top)| + |R \cap p;t^\top| + |R \cap t;p^\top| & \text{// by (C2), } R = R^\top \\
= |R \cap (s;t^\top \cup t;s^\top)| + |(R \cap p;t^\top) \cup (R \cap t;p^\top)| & \text{// 2nd auxiliary result} \\
= |(R \cap (s;t^\top \cup t;s^\top)) \cup (R \cap p;t^\top) \cup (R \cap t;p^\top)| & \text{// 3rd auxiliary result} \\
= |R \cap ((s \cup p);t^\top \cup t;(s \cup p)^\top)| &
\end{array}
$$

Correctness Proof: Cardinalities

Invariant

$$
\begin{aligned}
& |R \cap ((s \cup p);(s \cup p)^{\mathsf{T}} \cup t;t^{\mathsf{T}})| \\
&= |(R \cap (s;s^{\mathsf{T}} \cup t;t^{\mathsf{T}} \cup s;p^{\mathsf{T}} \cup p;s^{\mathsf{T}} \cup p;p^{\mathsf{T}})| \\
&= |(R \cap (s;s^{\mathsf{T}} \cup t;t^{\mathsf{T}})) \cup (R \cap s;p^{\mathsf{T}}) \cup (R \cap p;s^{\mathsf{T}})| && \text{// 1st aux. result} \\
&\leq |R \cap (s;s^{\mathsf{T}} \cup t;t^{\mathsf{T}})| + |R \cap s;p^{\mathsf{T}}| + |R \cap p;s^{\mathsf{T}}| && \text{// by (C3)} \\
&\leq |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R \cap s;p^{\mathsf{T}}| + |R \cap p;s^{\mathsf{T}}| && \text{// invariant} \\
&= |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R \cap p;s^{\mathsf{T}}| + |R \cap p;s^{\mathsf{T}}| && \text{// by (C2), } R = R^{\mathsf{T}} \\
&= |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R;p \cap s| + |R;p \cap s| && \text{// 1st aux. lemma} \\
&< |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R;p \cap t| + |R;p \cap t| && \text{// as } |R;p \cap s| < |R;p \cap t| \\
&= |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R \cap p;t^{\mathsf{T}}| + |R \cap p;t^{\mathsf{T}}| && \text{// aux. lemma} \\
&= |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |R \cap p;t^{\mathsf{T}}| + |R \cap t;p^{\mathsf{T}}| && \text{// by (C2), } R = R^{\mathsf{T}} \\
&= |R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})| + |(R \cap p;t^{\mathsf{T}}) \cup (R \cap t;p^{\mathsf{T}})| && \text{// 2nd auxiliary result} \\
&= |(R \cap (s;t^{\mathsf{T}} \cup t;s^{\mathsf{T}})) \cup (R \cap p;t^{\mathsf{T}}) \cup (R \cap t;p^{\mathsf{T}})| && \text{// 3rd auxiliary result} \\
&= |R \cap ((s \cup p);t^{\mathsf{T}} \cup t;(s \cup p)^{\mathsf{T}})|
\end{aligned}
$$

- not nice, but still (in)equational reasoning
  hence proof assistants can easily be used
- verification of postcondition is similar (but shorter)

Conclusion and Remarks

- verification of graph algorithms using cardinalities
- made use of point axiom $\mathsf{L}_{X\mathbf{1}} = \bigcup_{p \in \mathcal{P}_{\mathsf{L}_{X\mathbf{1}}}} p$
- made use of RelView (Berghammer et. al) to check proof invariants

Proof support and Proof Automation

- automated theorem provers (ATPs)
  - Prover9 (best for algebraic reasoning): no types
  - other have types, but difficult to encode heterogeneous RA
  - no (proper) support of intermediate lemmas
- Isabelle/HOL
  - excellent library for homogeneous RA [Str14]
  - no (proper) library for heterogeneous RA (Guttmann)
  - good connection to ATPs (via the Sledgehammer tool) allows proof automation
- Coq
  - good support for types
  - excellent library for (homogenous and heterogenous) RA [Pou]
  - lots of tactics available (decision procedures, normalisation ...)
  - cardinalities have been implemented (Stucke)
  - however, no tool such as sledgehammer

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Thank you

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