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From **imagination** to **impact**

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Algebras for (automatic) Verification of Graph Algorithms

Peter Höfner

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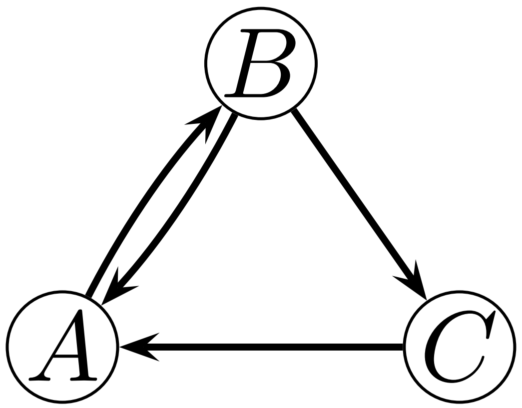
Motivation

- towards more automation in program verification
  - functional correctness
  - use algebra to improve proof automatisation
  - using pre/post conditions (Hoare-style reasoning)

- at the moment
  - look at ‘simple’ and well-known while programs (invariant proofs)
  - find ‘correct’/appropriate algebra
  - limited to algorithms where data structure can be modelled by algebra

Unweighted Graphs



$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{(A, B), (B, A), (B, C), (C, A)\}$$

- edges are relation between nodes
- relation algebra prime candidate
  - elements are sets of relations/Boolean matrices
  - offers operations for
    - sequential composition
    - set operations (union, intersection, complement)
    - transposition
    - finite iteration (Kleene star)
- well known, used for program verification

Warming Up: Reflexive–Transitive Closure

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input R
$C, v := \mathsf{I}, \mathsf{O};$

while $v \neq R;\mathsf{L}$ **do**
  let $p = point(R;\mathsf{L} \cap \overline{v});$
  $C, v := C \cup C;p;p^{\mathsf{T}};R;C \;,\; v \cup p$

od
return C

Warming Up: Reflexive-Transitive Closure

$\textbf{input } R$
$C, v := \mathsf{I}, \mathsf{O};$

$\textbf{while } v \neq R;\mathsf{L} \textbf{ do}$
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$\textbf{od}$
$\textbf{return } C$

deterministic function returning
a point from R, which was not considered
before

Warming Up: Reflexive-Transitive Closure

$$
\begin{array}{l}
\textbf{input } R \\
C, v := \mathsf{I}, \mathsf{O}; \\
\\
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\quad \textbf{let } p = point(R;\mathsf{L} \cap \overline{v}\,); \\
\quad C, v := C \cup C;p;p^{\mathsf{T}};R;C\ ,\ v \cup p \\
\\
\textbf{od} \\
\textbf{return } C \\
\{C = R^*\}
\end{array}
$$

deterministic function returning
a point from R, which was not considered
before

Warming Up: Reflexive-Transitive Closure

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$\{\text{True}\}$
input R
$C, v := \mathsf{I}, \mathsf{O};$

while $v \neq R;\mathsf{L}$ **do**
  let $p = point(R;\mathsf{L} \cap \overline{v});$
  $C, v := C \cup C;p;p^{\mathsf{T}};R;C \;,\; v \cup p$

od
return C
$\{C = R^*\}$

deterministic function returning
a point from R, which was not considered
before

Warming Up: Reflexive-Transitive Closure

$\{\mathrm{True}\}$
input R
$C, v := \mathsf{I}, \mathsf{O};$
$\{C = (R \cap v)^* \ \wedge\ v = v;\mathsf{L}\}$
while $v \neq R;\mathsf{L}$ **do**
  let $p = point(R;\mathsf{L} \cap \overline{v}\,);$
  $C, v := C \cup C;p;p^\top;R;C\ ,\ v \cup p$

od
return C
$\{C = R^*\}$

deterministic function returning
a point from R, which was not considered
before

Warming Up: Reflexive Transitive Closure

$$Inv_0(R, C, v) \;\Leftrightarrow\; C = (R \cap v)^*$$
$$Inv_1(v) \;\Leftrightarrow\; v = v \,;\mathsf{L}$$

- Proof: simple exercise?

Warming Up: Reflexive Transitive Closure

$$Inv_0(R, C, v) \;\Leftrightarrow\; C = (R \cap v)^*$$
$$Inv_1(v) \;\Leftrightarrow\; v = v\,;\mathsf{L}$$

- Proof: simple exercise?
- p is point $\;\Leftrightarrow\; p;\mathsf{L} = p \,\wedge\, \mathsf{L};p = \mathsf{L} \,\wedge\, p;p^\top \subseteq \mathsf{I}$

Warming Up: Reflexive Transitive Closure

$$Inv_0(R,C,v) \;\Leftrightarrow\; C = (R \cap v)^*$$
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- Proof: simple exercise?
- p is point $\;\Leftrightarrow\; p;\mathsf{L} = p \,\wedge\, \mathsf{L};p = \mathsf{L} \,\wedge\, p;p^\top \subseteq \mathsf{I}$

- Proof Automatisation
  (Prover9 or any other automated Theorem Prover)

Establishment	
$Inv_0(R,\mathsf{I},\mathsf{O}) \wedge Inv_1(\mathsf{O})$	0s
Post-Condition	
$v = R\,;\mathsf{L} \wedge Inv_0(R,C,v) \wedge Inv_1(v) \;\Rightarrow\; C = R^*$	0s
Maintainance	
$Inv_1(v) \wedge p$ is point $\wedge\, p \subseteq R\,;L \cap \overline{v} \;\Rightarrow\; Inv_1(v \cup p)$	1s
$Inv_0(R,C,v) \wedge p$ is point $\wedge\, p \subseteq R\,;L \cap \overline{v} \;\Rightarrow\; Inv_0(R, C \cup C\,;p\,;p^\top;R\,;C, v \cup p)$	-

Warming Up: Reflexive Transitive Closure

$$Inv_0(R, C, v) \;\Leftrightarrow\; C = (R \cap v)^*$$
$$Inv_1(v) \;\Leftrightarrow\; v = v\,;\mathsf{L}$$

- Proof: simple exercise?
- p is point $\;\Leftrightarrow\; p;\mathsf{L} = p \,\wedge\, \mathsf{L};p = \mathsf{L} \,\wedge\, p;p^\top \subseteq \mathsf{I}$

- Proof Automatisation
  (Prover9 or any other automated Theorem Prover)

Establishment	
$Inv_0(R, \mathsf{I}, \mathsf{O}) \wedge Inv_1(\mathsf{O})$	0s
Post-Condition	
$v = R\,;\mathsf{L} \wedge Inv_0(R, C, v) \wedge Inv_1(v) \;\Rightarrow\; C = R^*$	0s
Maintenance	
$Inv_1(v) \wedge p \text{ is point} \wedge p \subseteq R\,;L \cap \overline{v} \;\Rightarrow\; Inv_1(v \cup p)$	1s
$Inv_0(R, C, v) \wedge p \text{ is point} \wedge p \subseteq R\,;L \cap \overline{v} \;\Rightarrow\; Inv_0(R, C \cup C\,;p\,;p^\mathsf{T}\,;R\,;C, v \cup p)$	0s

+ 3 properties about Kleene star

More Examples

input R

$S, v := \mathsf{I}, \mathsf{O};$
while $v \neq \mathsf{L}$ **do**

  let $p = point(\,\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}}\,) ; \overline{v}}\,);$
  $S, v := S \cup v ; p^{\mathsf{T}}, v \cup p$

od
return S

More Examples

- Topological Sorting

$$
\begin{array}{l}
\textbf{input } R \\
\\
S, v := \mathsf{I}, \mathsf{O}; \\
\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\\
\qquad \textbf{let } p = point(\,\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}}\,);\overline{v}}\,); \\
\qquad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
\\
\textbf{od} \\
\textbf{return } S
\end{array}
$$

More Examples

- Topological Sorting

$$
\begin{array}{l}
\textbf{input } R \\
\{R;R^* = \mathsf{O}\} \\
S, v := \mathsf{I}, \mathsf{O}; \\
\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\quad \textbf{let } p = point(\,\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}}\,);\overline{v}}\,); \\
\quad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
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More Examples

- Topological Sorting

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\begin{array}{l}
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\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\qquad \textbf{let } p = point(\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}});\overline{v}}); \\
\qquad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
\textbf{od} \\
\textbf{return } S \\
\{R \subseteq S \wedge \mathsf{I} \subseteq S \wedge S;S \subseteq S \wedge S \cap S^{\mathsf{T}} \subseteq \mathsf{I} \wedge S \cup S^{\mathsf{T}} = \mathsf{L}\}
\end{array}
$$

More Examples

- Topological Sorting

$$
\begin{array}{l}
\textbf{input } R \\
\{R;R^* = \mathsf{O}\} \\
S, v := \mathsf{I}, \mathsf{O}; \\
\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\{\mathsf{I} \subseteq S \ \wedge \ S;S \subseteq S \ \wedge \ S \cap S^{\mathsf{T}} \subseteq S \ \wedge \ S \cup S^{\mathsf{T}} = v;v^{\mathsf{T}} \cup \mathsf{I} \wedge \\
\\
\quad \textbf{let } p = point(\, \overline{v} \ \cap \ \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}}\,);\, \overline{v}}\ ); \\
\quad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
\\
\textbf{od} \\
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\{R \subseteq S \ \wedge \ \mathsf{I} \subseteq S \ \wedge \ S;S \subseteq S \ \wedge \ S \cap S^{\mathsf{T}} \subseteq \mathsf{I} \ \wedge \ S \cup S^{\mathsf{T}} = \mathsf{L}\}
\end{array}
$$

More Examples

- Topological Sorting

$$
\begin{array}{l}
\textbf{input } R \\
\{R;R^* = \mathsf{O}\} \\
S, v := \mathsf{I}, \mathsf{O}; \\
\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\{\mathsf{I} \subseteq S \ \wedge \ S;S \subseteq S \ \wedge \ S \cap S^{\mathsf{T}} \subseteq S \ \wedge \ S \cup S^{\mathsf{T}} = v;v^{\mathsf{T}} \cup \mathsf{I} \wedge \\
\quad v;\mathsf{L} \subseteq v \ \wedge \ S;v \subseteq v \ \wedge \ R \cap v;v^{\mathsf{T}} \subseteq S \ \wedge \ R;v \subseteq v\} \\
\qquad \textbf{let } p = point(\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}});\overline{v}}); \\
\qquad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
\\
\textbf{od} \\
\textbf{return } S \\
\{R \subseteq S \ \wedge \ \mathsf{I} \subseteq S \ \wedge \ S;S \subseteq S \ \wedge \ S \cap S^{\mathsf{T}} \subseteq \mathsf{I} \ \wedge \ S \cup S^{\mathsf{T}} = \mathsf{L}\}
\end{array}
$$

More Examples

- Topological Sorting

Proof is fully automatic (incl. termination)

$$
\begin{array}{l}
\textbf{input } R \\
\{R;R^* = \mathsf{O}\} \\
S, v := \mathsf{I}, \mathsf{O}; \\
\textbf{while } v \neq \mathsf{L} \textbf{ do} \\
\{\mathsf{I} \subseteq S \wedge S;S \subseteq S \wedge S \cap S^{\mathsf{T}} \subseteq S \wedge S \cup S^{\mathsf{T}} = v;v^{\mathsf{T}} \cup \mathsf{I} \wedge \\
\quad v;\mathsf{L} \subseteq v \wedge S;v \subseteq v \wedge R \cap v;v^{\mathsf{T}} \subseteq S \wedge R;v \subseteq v\} \\
\qquad \textbf{let } p = point(\overline{v} \cap \overline{(R^{\mathsf{T}} \cap \overline{\mathsf{I}});\overline{v}}); \\
\qquad S, v := S \cup v;p^{\mathsf{T}}, v \cup p \\
\\
\textbf{od} \\
\textbf{return } S \\
\{R \subseteq S \wedge \mathsf{I} \subseteq S \wedge S;S \subseteq S \wedge S \cap S^{\mathsf{T}} \subseteq \mathsf{I} \wedge S \cup S^{\mathsf{T}} = \mathsf{L}\}
\end{array}
$$

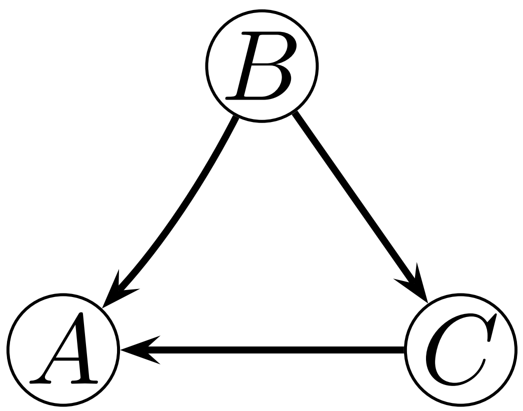
More Examples

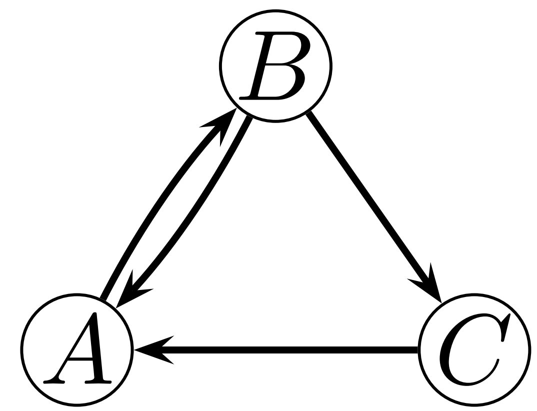
- Matching Algorithm
- Node Colouring
- …

- **Relation algebra seems to be well suited for most (all?) graph problems**

Subtree

- natural order: $\subseteq$

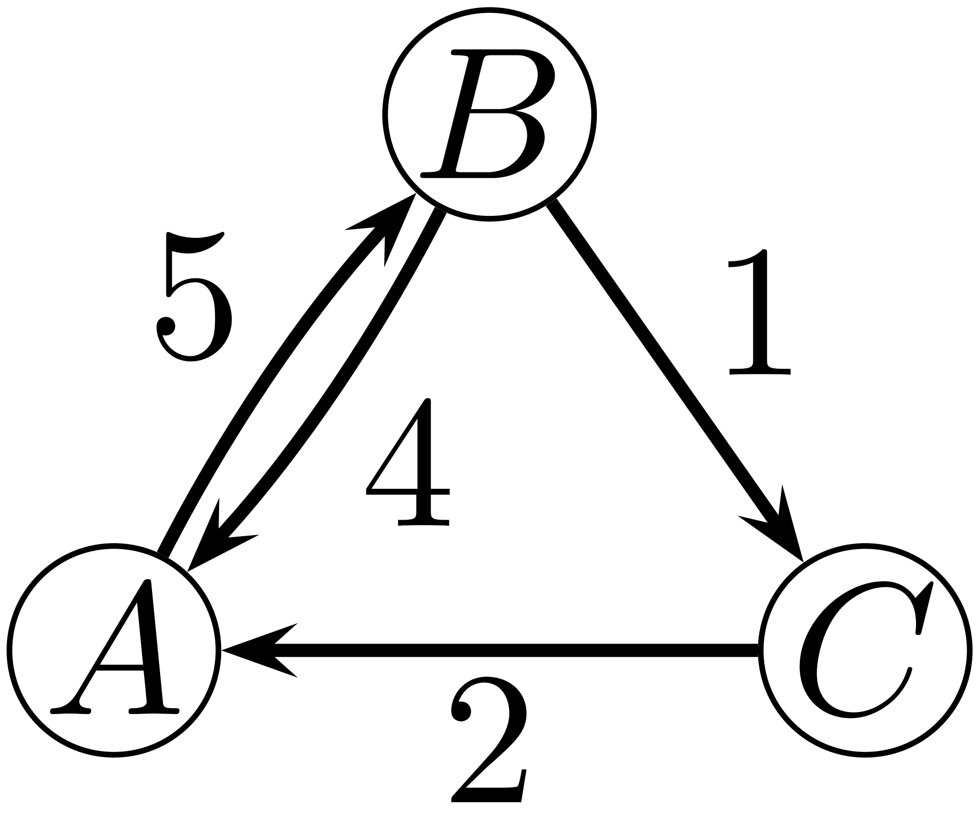




$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\{ (B,A), (B,C), (C,A)\} \subseteq \{(A,B), (B,A), (B,C), (C,A)\}$$

Weighted Graphs



Algebras for Weighted Graphs

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- Matrices over Min-Plus-Algebra (and variants)
  - algorithms such as Dijkstra and Floyd-Warshall
- Routing Algebra
  - developed for Mesh Protocols
    (see IFIP 2.1 Reisensburg)
- Other algebras: Max-Plus, Max-Min, Min-Max, …

Min-Plus Algebra

- Choice: Take path with smaller weight
- Path Composition: Addition
- Kleene star: $n^* = \min_{i \geq 0}(\sum_{j=0}^{i} n) = \min(0, n, 2n, \dots) = 0$
- $(\mathbb{N} \cup \{\infty\}, \min, +, \infty, 0, ^*)$ forms a Kleene algebra
    - no intersection, no complement
    - no transposition
    - natural order defined as usual

$$m \sqsubseteq n \Leftrightarrow \min(m, n) = n \Leftrightarrow n \leq m$$

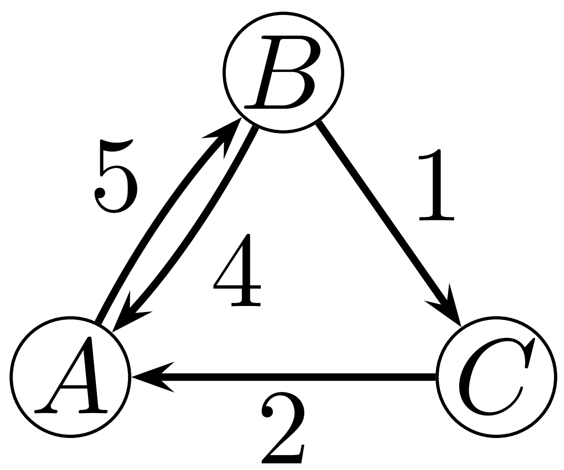
- Theorem:
  Matrices over Kleene algebras are Kleene algebras
    - natural order is defined point-wise

Problems

- Is this algebra as suitable and flexible as relation algebra?

Reflexive-Transitive Closure

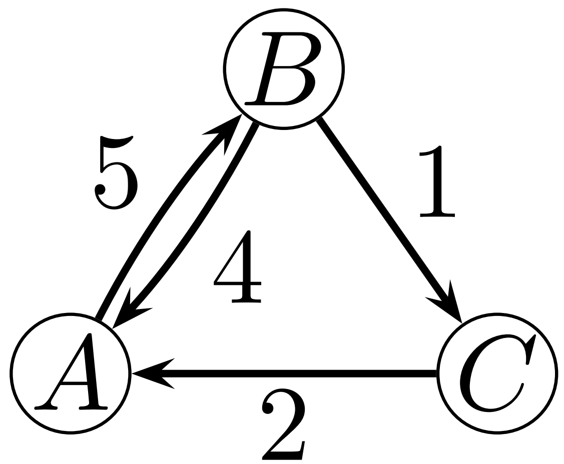
- all-shortest paths



$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

Reflexive-Transitive Closure

- all-shortest paths

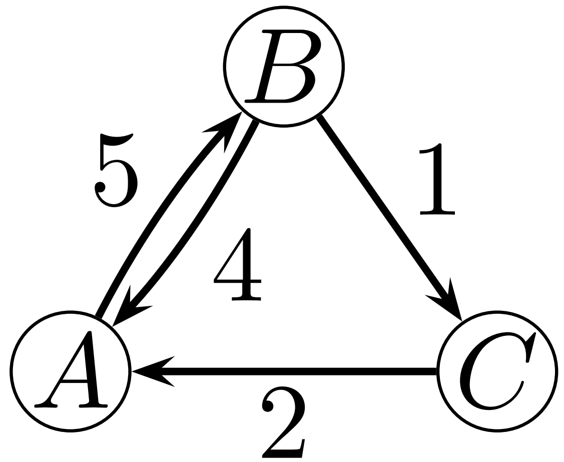


$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

$$G^* = \begin{pmatrix} 0 & 5 & 6 \\ 3 & 0 & 1 \\ 2 & 7 & 0 \end{pmatrix}$$

Reflexive-Transitive Closure

- all-shortest paths



$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

- How to calculate the star
  - classical matrix decomposition (cf. Kozen)
  - algorithm from above ?

Reflexive Transitive Closure

$$
\begin{array}{l}
\textbf{input } R \\
C, v := \mathsf{I}, \mathsf{O}; \\
\\
\textbf{while } v \neq R;\mathsf{L} \textbf{ do} \\
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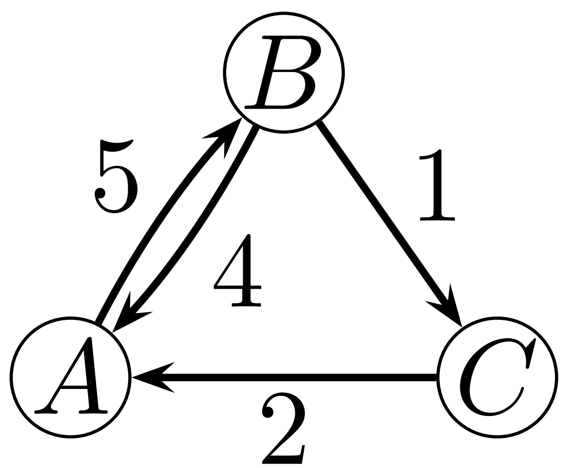
\quad C, v := C \cup C;p;p^{\mathsf{T}};R;C \;,\; v \cup p \\
\\
\textbf{od} \\
\textbf{return } C
\end{array}
$$

Reflexive-Transitive Closure

- all-shortest paths



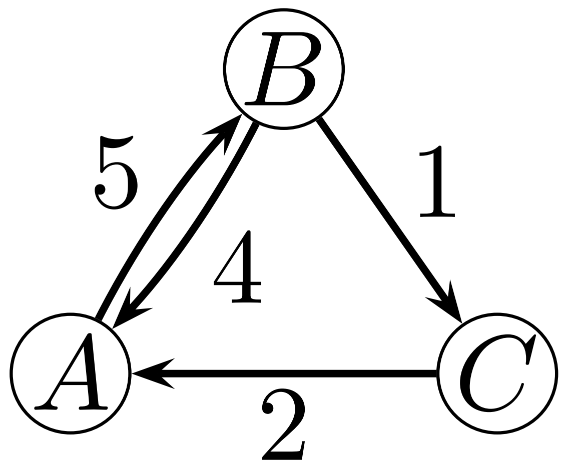
$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

- How to calculate the star
  - classical matrix decomposition (cf. Kozen)
  - algorithm from above
    - problem: what is a point

$$p;\mathsf{L} = p \;\wedge\; \mathsf{L};p = \mathsf{L} \;\wedge\; p;p^\top \subseteq \mathsf{I}$$

Reflexive-Transitive Closure

- all-shortest paths



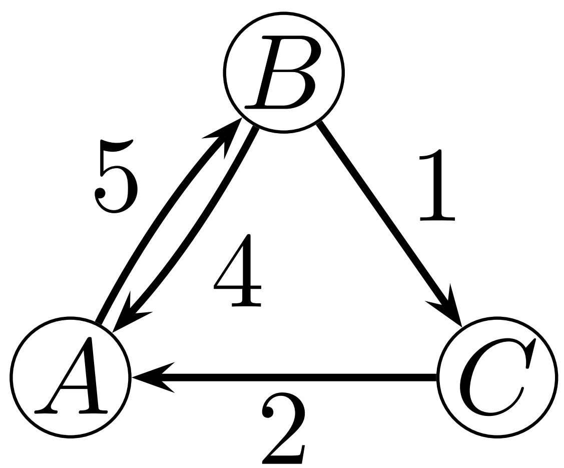
$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

- How to calculate the star
  - classical matrix decomposition (cf. Kozen)
  - algorithm from above
    - problem: what is a point

$$p \cdot \top = p \;\wedge\; \top \cdot p = \top \;\wedge\; p \cdot p^{\top} \sqsubseteq \mathsf{Id}$$

Reflexive-Transitive Closure

- all-shortest paths



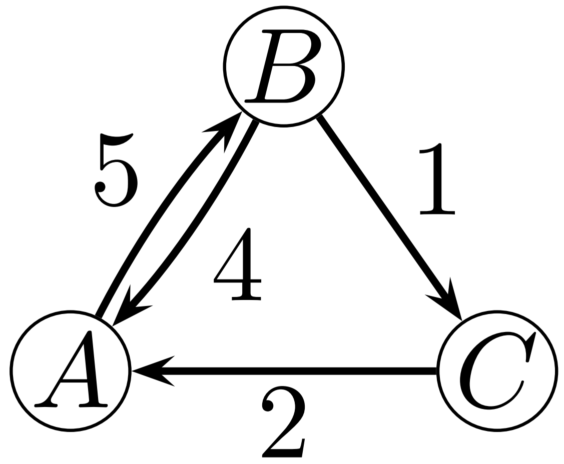
$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

- How to calculate the star
  - classical matrix decomposition (cf. Kozen)
  - algorithm from above
    - problem: what is a point

$$p \cdot \top = p \wedge \top \cdot p = \top \wedge \cancel{p \cdot p^\top} \sqsubseteq \mathsf{Id}$$

Reflexive-Transitive Closure

- all-shortest paths



$$G = \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

- How to calculate the star
  - classical matrix decomposition (cf. Kozen)
  - algorithm from above
    - points can be characterised via atomic test elements (every Kleene algebra can be equipped with a test algebra — no details in this talk)

Example: Prim's algorithm

input G, v
$\{G \text{ symmetric}\}$
$U, T := v, 0;$
while $U \neq \mathsf{Id}$ **do**
$\{T \text{ is minimal spanning tree in } U \cdot G \cdot U\}$
  let e edge with minimal weight from U to $\neg U$
  $U, T := U + \text{source of } e \ , \ T + e$
od
return T
$\{T \text{ is minimal spanning tree}\}$

Example: algorithm to compute Spanning Tree

input G, v

$\{G \text{ symmetric}\}$

$U, T := v, 0;$

while $U \neq \mathsf{Id}$ **do**

$\{T \text{ is spanning tree in } U \cdot G \cdot U\}$

    let e edge from U to $\neg U$

    $U, T := U + \text{source of } e\ ,\ T + e$

od

return T

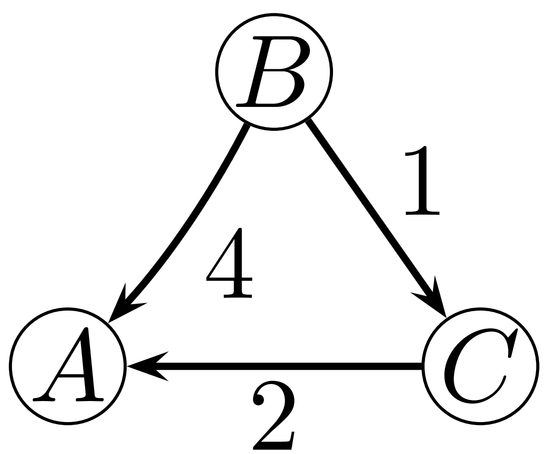
$\{T \text{ is spanning tree}\}$

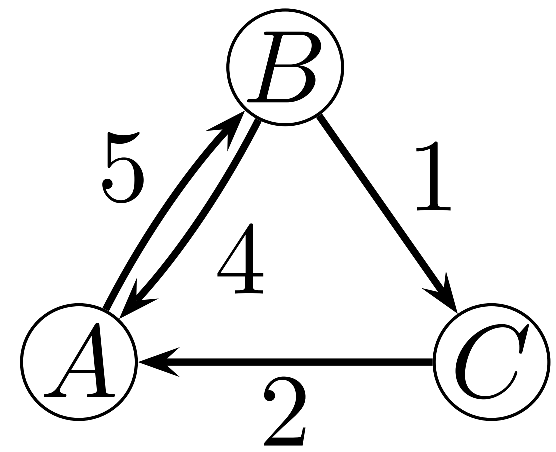
Spanning Tree

- T is spanning tree of G
  - T is tree (injective, reaches everything)
  - T is *subtree* of G

Subtree

- natural order: $\sqsubseteq$

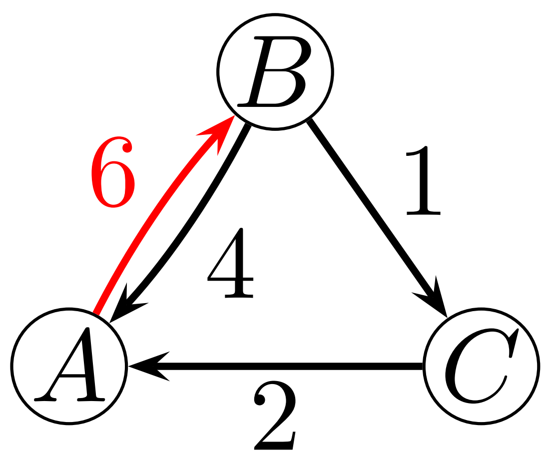


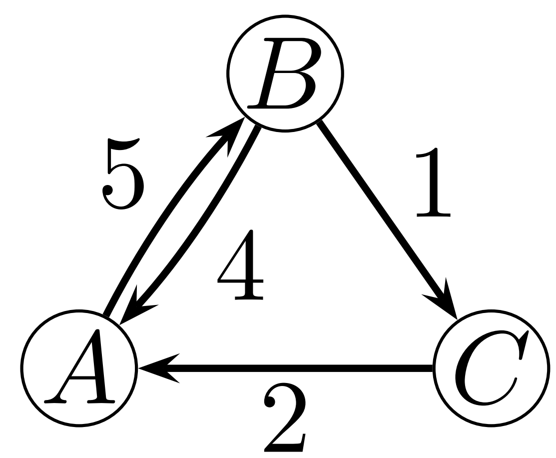


$$\begin{pmatrix} \infty & \infty & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix} \sqsubseteq \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

Subtree

- natural order: $\sqsubseteq$





$$\begin{pmatrix} \infty & \color{red}{6} & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix} \sqsubseteq \begin{pmatrix} \infty & 5 & \infty \\ 4 & \infty & 1 \\ 2 & \infty & \infty \end{pmatrix}$$

`

- Relation algebra (set model): $(\wp(V \times V), \cup, \cap, \ldots)$
- "pseudo" multigraphs (Matrices with sets as entries)
  - $(\wp(V \times \mathbb{N} \times V), \cup, +_{\text{join}}, \emptyset, V \times \{0\} \times V, {}^*)$
    forms Kleene algebra, where $+_{\text{join}}$ is point-wise operation

$$(u, m, v) +_{\text{join}} (w, n, x) = \begin{cases} (u, m+n, x) & \text{if } v = w \\ \text{undefined} & \text{otherwise} \end{cases}$$

  - $(\wp(V \times \mathbb{Z} \times V), \cup, +_{\text{join}}, \emptyset, V \times \{0\} \times V, {}^*)$
    can be turned into a Relation algebra

$$(u, m, v)^{\top} = (v, -m, u)$$

- why not real multi graphs?
  no natural order

Back to Spanning tree

input G, v
$\{G \text{ symmetric}\}$
$U, T := v, 0;$
while $U \neq \mathsf{Id}$ **do**
$\{T \text{ is spanning tree in } U \cdot G \cdot U\}$
    let e edge from U to $\neg U$
    $U, T := U + \text{source of } e \;,\; T + e$
od
return T
$\{T \text{ is spanning tree}\}$

Back to Spanning tree

$\textbf{input } G, v$
$\{G \text{ symmetric}\}$
$U, T := v, 0;$
$\textbf{while } U \neq \mathsf{Id} \textbf{ do}$
$\{T \leq U \cdot G \cdot U \ \wedge \ \mathsf{range}(v \cdot T^{+}) = U\}$
$\quad \textbf{let } e \text{ edge with } e \leq U \cdot G \cdot \neg U \,, \ldots;$
$\quad U, T := U + \mathsf{source}(e) \ , \ T + e$
$\textbf{od}$
$\textbf{return } T$
$\{T \text{ is spanning tree}\}$

Spanning tree

- Correctness can be shown similar to the above examples
  - in all three models
  - straight-forward (full automatic if isotonicity laws are added)
  - source and range can be defined via algebraic operations (tests, domain, codomain)

- But: How to characterise minimality?

Problem: Minimality

- Easy if additional weight-function on top
  - model dependent, requires specific axioms for functions…
  - could be performed on relations only
  - but seems not to be the best way

- can we integrate minimality into algebra?
  - how to access the weights?
  - in $(\wp(V \times \mathbb{N} \times V), \cup, +_{\mathrm{join}}, \emptyset, V \times \{0\} \times V, ^{*})$
    one can at least compare edges

$$e_1 \text{ preferred over } e_2 \;\Leftrightarrow\; \top \cdot e_1 \cdot \top \leq \top \cdot e_2 \cdot \top$$

Summary

- aim at more automation for program verification
  - "black-box" approach
  - any ATP/ITP system should be fine
- focus on graph algorithms
- suitable algebras
  - unweighted graphs: relation algebra
  - shortest paths: min-plus algebra
    (building graphs)
  - spanning trees: ???
    (subtrees)
  - max-plus algebra, max-min algebra …

- weighted graphs need several algebraic models
  (hopefully all based on same algebra)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From **imagination** to **impact**