Towards a Representation Theorem for Coloring Algebra

Peter Höfner
(joint work with B. Möller and A. Zelend)

Amsterdam
September 12, 2012

Motivation

- feature-orientation
  - feature-oriented software development (FOSD)
  - feature-oriented domain analysis (FODA)
  - feature-oriented programming (FOP)
- main idea
  - level-based design,
    i.e., the idea that each program can be successively built up by adding more and more levels
  - feature: increment in functionality or in the software development.

Motivation

- applications
  - network protocols, data structures, software product lines ...
- support by software systems
  - Ahead Tool Suite (Batory)
  - Colored IDE (Kästner)
  - Feature House (Apel, Lengauer)
  - GenVoca (Batory)
- several case studies

Motivation

- little understanding of the structures (mathematics) behind
- algebraic models
  - Feature Algebra (Apel, Kästner, Lengauer, Möller)
    - model for FeatureHouse
    - based on common ideas of FO (introductions, refinements and quantification)
    - central element: feature structure forrest
  - **Coloring Algebra (Batory, Höfner, Kim)**
    - model inspired by Colored IDE
    - models feature composition and interaction
    - standard model use variation points
- goals
  - understand underlying algebraic structures
  - gain better understanding for FO

Coloring Algebra – Example

- example (fire and flood control)

$$fire \times flood \;=\; (fire \cdot flood) + fire + flood$$

  - feature composition (combine properties)
    disjoint union of features
  - feature interaction (repair)
    set of changes that are needed to make features work together
  - full interaction (combine single features with their repairs)

  - $fire \cdot flood$ indicates an interaction between the features $fire$ and $flood$ and resolves their conflicts

Coloring Algebra – Composition

- feature composition $+$
- adding a feature twice removes it (similar to a switch)

- Definition:
  $(F, +, 0)$ is a commutative group that satisfies involution

$$f + (g + h) = (f + g) + h$$
$$f + g = g + f$$
$$f + 0 = f$$
$$f + f = 0$$

Coloring Algebra – Interaction

- feature interaction $\cdot$
- describes conflicts and “repairs”

- Definition:
  $(F, \cdot, 0)$ is a commutative group that satisfies involution

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h$$
$$f \cdot g = g \cdot f$$
$$f \cdot 0 = 0$$
$$f \cdot f = 0$$

Coloring Algebra

- Definition:
  a *Coloring Algebra* (CA) is a structure $(F, +, \cdot, 0)$ such that $(F, +, 0)$ is a (commutative) involutive group and $(F, \cdot)$ is a commutative, involutive semigroup.
  Moreover, interaction distributes over composition

- hence, a CA is a special ring

- for a given CA, full composition is defined as

$$f \times g \;=\; (f \cdot g) + f + g$$

1/2 Representation Theorem

- feature composition
  - every element has order 2 (involution)
  - any finite 2-group is a power of $\mathbb{Z}_2$
  - By the Kronecker Basis Theorem there is exactly one finite model satisfying the axioms for the cardinalities 2, 4, 8, ...
    (no model otherwise)

- Representation Theorem
  Every finite algebra satisfying the axioms for feature composition is isomorphic to a model that can be obtained by using symmetric difference on a power set of a finite set.

  (proof can be achieved by a constructing a generating system)

More Models (generic)

- assume a set B of *base colors.* Then $(2^B, \Delta, \emptyset)$ satisfies the axioms of feature composition, where Δ denotes symmetric difference

$$M \,\Delta\, N \;=\; (M \cup N) - (M \cap N)$$

- a first generic model of CA can be defined by

$$M \cdot N \;=\; \emptyset$$

Towards A Full Representation Theorem

- feature composition well-known
  - generic models for composition
- feature interaction not understood yet
  - more models due to variability
  - no generic models known
  - sequel: towards a better understanding of feature interaction
    vision: full representation theorem

Base Colors

- from now we use the generic model for feature composition (Δ)
- free use of set-theoretic concepts and operations

- every element of a finite CA is finitely generated, i.e., there is a set B of *base colors*
- general: an element is base if it is isomorphic to a singleton set (atom) of the generic model
- due to distributivity the definition of interaction can be reduced to the interaction of base colors

Small Models (generated)

- generated models (Mace4)

#base colors/ #colors	#interact. (up to iso.)	# CA (up to iso.)
1/2	1	1
2/4	2	1
3/8	557	2
4/16		2

Models (general)

- assume a semigroup $(B, \circ)$
  including a special element e, satisfying
  $$e \circ a = e = a \circ e$$

- feature interaction can be defined as
  $$f \cdot g \;=\; \mathop{\Delta}_{a \in f} \mathop{\Delta}_{b \in g} \iota(a \circ b)$$
  where $\iota : B \to 2^B$ is given by $\iota(e) = \emptyset$ and $\iota(b) = \{b\}$
  (for $b \neq e$)

Representation Conjecture

- $(2^B, \Delta, \cdot, \emptyset)$ forms a Coloring Algebra
- Conjecture: These are all models up to isomorphism (proof missing)
- in real applications it is useful to define B as power set of even smaller units (2^P)

Feature Interaction

- operation $\circ$ gives flexibility
- however, the axioms imply a clear structure for interactions

Feature Interaction (first consequences)

- Lemma:
  a repair cannot introduce new conflicts

$$f \cdot g = h \;\Rightarrow\; f \cdot h = 0$$

- Lemma:
  a repair does not delete one of its components entirely

$$f \neq 0 \;\Rightarrow\; f \cdot g = 0$$

Feature Interaction (first consequences)

- Lemma:
  colors cannot repair each other in “cycles”
  - no non-trivial color is its own repair
$$f \cdot g = f \;\Rightarrow\; f = 0$$
  - repairs are mutually exclusive
$$f \cdot h_1 = g \wedge g \cdot h_2 = f \;\Rightarrow\; f = 0$$
  - the first part can be extended to finite chains
$$f \cdot h_1 = h_2 \wedge \Big(\bigwedge_{i=1}^{n} h_{3i-1} \cdot h_{3i} = h_{3i+1}\Big) \wedge h_{3n+1} \cdot h_{3n+2} = f \;\Rightarrow\; f = 0$$

Feature Interaction (first consequences)

- absence of cycles makes the divisibility relation w.r.t. interaction into a strict partial order (on non-empty colors)

$$f < g \;\; \Leftrightarrow_{df} \;\; f, g \in F - \{0\} \, \wedge \, \exists h \in F : f \cdot h = g$$

- Lemma:
  neither composition nor interaction is isotone w.r.t. $<$

Interaction Equivalence and Ideals

- group colors according to their behaviour under interaction

$$f \sim g \;\; \Leftrightarrow_{df} \;\; \forall h : f \cdot h = g \cdot h$$

- $\sim$ is an equivalence relation,
  equivalence classes are denoted by

$$[f] \;\; =_{df} \;\; \{g \,|\, f \sim g\}$$

- Lemma:
  an element of $F - \{0\}$ is an annihilator iff it is maximal w.r.t. $<$

  for finite $F \neq \{0\}$ there is at least one maximal element in $F - \{0\}$ and hence a non-zero annihilator

Interaction Equivalence and Ideals

- Lemma:
  the set $[0]$ of annihilators forms a subtractive ring ideal
  - closed under composition
  - closed under interaction
  - $f \in [0] \wedge f + g \in [0] \Rightarrow g \in [0]$

Interaction Equivalence and Ideals

- Lemma:
  composition is cancellative w.r.t. $\sim$

$$f + h \sim g + h \;\Leftrightarrow\; f \sim g$$

  $\sim$ is a congruence

Towards a Representation Theorem

- generating systems w.r.t. interaction $\cdot$
- Lemma:
  let G be a minimal generating system. Then no two distinct elements of G can be related by $\sim$

Towards a Representation Theorem

- Theorem:
  the elements of G form a system of representatives for the equivalence classes of $\sim$
  the set of equivalence classes can be made into a quotient semiring by defining

$$[f] + [g] =_{df} [f + g]$$
$$[f] \cdot [g] =_{df} [f \cdot g]$$

Towards a Representation Theorem

- Conjectures
  - the interaction (repair) of two base colors is *always* a base color
  - the number of possible CAs can be determined by the number of semigroups satisfying the additional annihilation requirements
  - the elements $f \in 2^N$ form a system of representatives for equivalence classes, where N is the set of all non-annihilating base colors

References

- Peter Höfner, Bernhard Möller, Andreas Zelend: Foundations of Coloring Algebra with Consequences for Feature-Oriented Programming. In: W. Kahl and T.G. Griffin (Eds.): Relational and Algebraic Methods in Computer Science (RAMiCS 2012), LNCS 7560, pp. 33–49. Springer (2012)

- Don Batory, Peter Höfner, Jongwook Kim: Feature Interactions, Products, and Composition. In: 10th ACM International Conference on Generative Programming and Component Engineering (GPCE 2011), pp. 13–22. ACM Press (2011)

From **imagination** to **impact**

NICTA

From **imagination** to **impact**