Lazy Semiring Neighbours and some Applications

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Motivation

- interval logics are used for specification and verification of safety properties of reactive systems
- they cannot express properties like (unbounded) liveness:

  "eventually there will be an interval where ϕ holds"
- therefore Neighbourhood Logic (NL) [ZhouHansen96]

Deficiencies of Neighbourhood Logic

- complex expressions
- reasoning difficult (too many quantifiers)
- refer to single intervals
- cannot handle intervals of infinite length
  (necessary for reactive and hybrid systems)

Previous Work

- sets of intervals form Kleene algebra [Höfner03]
- NL embedded into Kleene algebra with domain [Höfner06]
  - quantifiers eliminated
  - calculations on sets of intervals possible
  - neighbours via domain/codomain
  - some NL-axioms can be dropped
  - iteration added
- question: how to handle infinite intervals ?

Results

- NL adapted to weak and lazy Kleene algebras [vonWright00, Möller04]
- NL expanded
  - handling of infinite intervals
  - most properties still hold
- connection to CTL*
- adaptation to reactive and hybrid systems

Outline

- From Neighbourhood Logic to Semirings
  - Neighbourhood Logic
  - embedding into Kleene algebra
  - discussion
- Adding infinity to NL
- NL, CTL* and hybrid systems
- Outlook

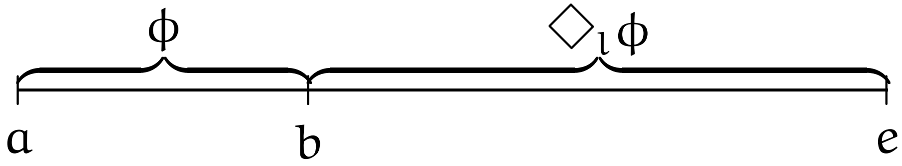
From Neighbourhood Logic to Semirings

about NL:

- purpose: reasoning about single time intervals
- chop-based interval temporal logics, like ITL and IL, cannot express all desired properties
- main idea [ZhouHansen96]:
  extend with ***left*** and ***right neighbourhoods***

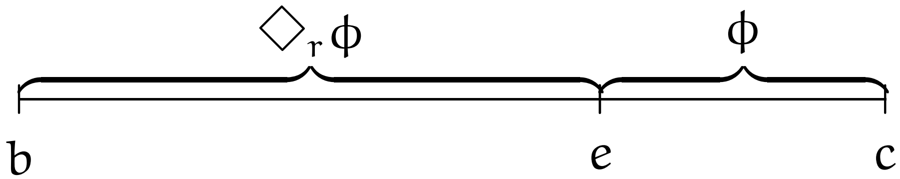
Neighbourhoods

left neighbourhood: $\Diamond_l \phi$



$\Diamond_l \phi$ holds on $[b, e]$ iff ϕ holds on some $[a, b]$

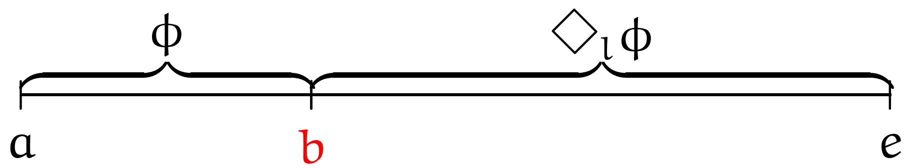
right neighbourhood: $\Diamond_r \phi$



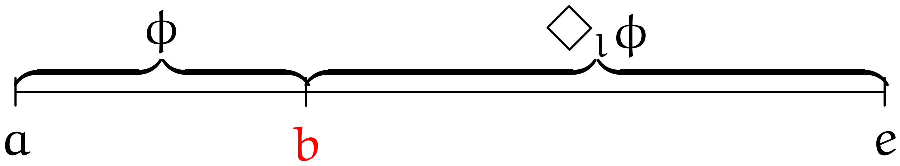
$\Diamond_r \phi$ holds on $[b, e]$ iff ϕ holds on some $[e, c]$

Properties

- *expanding* modalities
- neighbours only depend on *contact points*



- similar to sequential composition



- b is starting point of $[b, e]$: $[b, b] = \mathtt{dom}\ [b, e]$
- b is ending point of $[a, b]$: $[b, b] = \mathtt{ran}\ [a, b]$
- use Kleene algebra with domain [DesharnaisMöllerStruth03] with sets of intervals as elements

Definition

left neighbour: $x \leq \Diamond\!\!\!\!\!{\scriptstyle n}_l\, y \quad \Leftrightarrow_{df} \quad \mathtt{ran}\ x \leq \mathtt{dom}\ y$

right neighbour: $x \leq \Diamond\!\!\!\!\!{\scriptstyle n}_r\, y \quad \Leftrightarrow_{df} \quad \mathtt{dom}\ x \leq \mathtt{ran}\ y$

Theorem

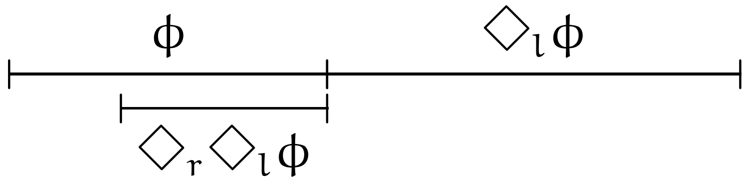
Let $[\![\phi]\!]$ *be the set of all intervals where* ϕ *holds. Then*

$$\Diamond_r \phi \textit{ holds on } x \quad \Leftrightarrow \quad x \leq \circledcirc_l [\![\phi]\!]$$

$$\Diamond_l \phi \textit{ holds on } x \quad \Leftrightarrow \quad x \leq \circledcirc_r [\![\phi]\!]$$

More Neighbours (briefly)

- simplifications of combinations, i,e, $\Diamond_r \Diamond_l \phi$:



  - common endpoints
  - yields equations similar to neighbours
  - *right boundary*: $x \leq \Diamond\!\!\!\!\!\!{\scriptstyle b}\;_r y$
- box operators also possible:

  "$\Box_l \phi$ holds on $[b, e]$ iff ϕ holds on **all** $[a, b]$"
- novel box operators of combinations

Results

- underlying structure yields
  - de Morgan dualities
  - Galois connections like

    $\langle n \rangle_l x \leq y \Leftrightarrow x \leq [n]_r y \quad$ and $\quad \langle n \rangle_r x \leq y \Leftrightarrow x \leq [n]_l y$
  - rich calculus for free
- simplifying NL
  - some NL-axioms can be dropped
  - additional box operators introduced
  - many properties follow from Galois connections
  - explicit expressions for neighbours, e.g., $\langle n \rangle_l y = \top \cdot \mathrm{dom}\ y$
- almost all results of NL can be lifted to semirings

Adding infinity to NL

- NL cannot handle intervals like $[a, \infty)$
- idea: shift to Kleene algebra without right-strictness
  - compositions of infinite intervals becomes
    $[a, \infty)\,;[b, c] \;=\; [a, \infty)$
- *lazy Kleene algebra*:
  no right-distributivity and right-strictness
  codomain not dual to domain

Properties

- neighbours defined as before
- distinguish finite and infinite parts of an element
  - finite elements form Kleene algebra;
    all properties hold
  - infinite elements loose properties
    e.g. right neighbour of an unbounded interval

  - all elements are right neighbours of infinite elements
    since $\mathsf{ran}\ [a, \infty) = 1$
  - and vice versa

Results

- theory adapted to weak and lazy Kleene algebra
- some properties are lost (e.g., due to codomain)
- only one kind of Galois connection
- NL expanded by intervals with infinite length
  - NL can now handle infinite traces
  - NL can be used for reactive and hybrid systems

Not Only For Neighbourhood Logic

neighbours also occur in

- CTL*
  - branching time logic [Emerson91]
  - algebraic version [MöllerHöfnerStruth06]

$$
\begin{array}{lclcl}
[\![\mathsf{E}\varphi]\!] & = & \mathtt{dom}\ [\![\varphi]\!] \cdot \top & = & \langle b\rangle_{\iota}[\![\varphi]\!] \\
[\![\mathsf{A}\varphi]\!] & = & \neg\mathtt{dom}\ (\overline{[\![\varphi]\!]}) \cdot \top & = & [b]_{\iota}[\![\varphi]\!]
\end{array}
$$

- hybrid systems

Conclusion and Outlook

- expanded NL by infinity
- connection to reactive and hybrid systems
- connection to CTL*
- knowledge transfer between different frameworks

- apply NL to the algebra of hybrid systems [HöfnerMöller04]
- especially case studies