An Algebra of Hybrid Systems

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"A world full of computers which you can't understand, can't fix and can't use [because it is controlled by inaccessible proprietorial software] is a world controlled by machines."

Eben Moglen

Hybrid Systems

Definition

Hybrid systems are heterogeneous systems characterised by the interaction of *discrete* and *continuous* dynamics.

Applications

- (air-)traffic controls / traffic management
- chemical and biological processes
- automated manufacturing
- Biohybrid Systems:
  Fraunhofer Institute for Biomedical Engineering (IBMT)
- standard example: leaking gas burner, (generalized) railroad crossing
- ...

Kinds of Systems

Transformational Systems
determine a function

Reactive Systems
interact with environment

Real–Time Systems
have to produce results within a certain amount of time

Hybrid Systems
discrete and continuous dynamics

source: University of Oldenburg

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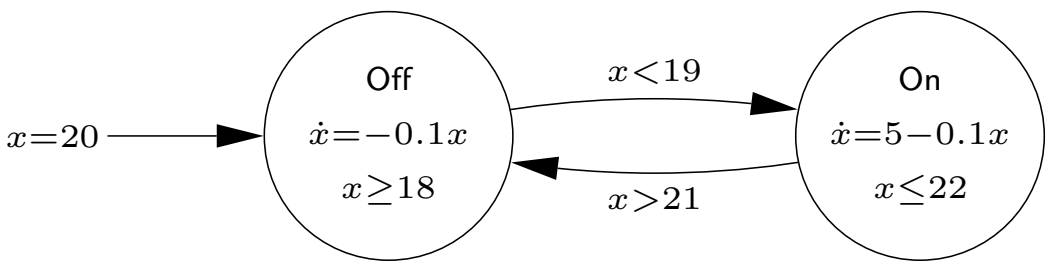
less than 1% of all processors are in PCs;
more than 98% are controllers of hybrid systems

Hybrid Automata

- most common representation type for hybrid systems
- widely popular for designing and modelling
- similar to finite state machines
- states describe continuous dynamics
- edges describe discrete behaviours

Example

Gas Burner:



Hybrid Automata

(Dis-)Advantages

- easy to construct/understand
- growing fast and becoming unreadable
- nearly impossible to check liveness or safety
  (only done partly for a small class of hybrid systems)
- nearly no software-tools available

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Question

(how) are hybrid automata related to X-Machines?

Towards an Algebra of Hybrid Systems

Idea

connection between finite state machines and regular languages

is a similar move possible

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Questions

- what are possible elements
- how to describe discrete and continuous behaviour
- how to describe infinity
  (interaction on an on-going, nearly never-ending basis)
- how to compose elements
- how to choose between elements

Towards an Algebra of Hybrid Systems

Possible Answers

- elements are trajectories
- continuous behaviour is described by the flow functions
- discrete behaviours are e.g. jumps in the function
- algebra is based on sets of trajectories
- weak Kleene algebra allows modelling infinite elements

(based on Sintzoff, Henzinger, Davoren, Lynch)

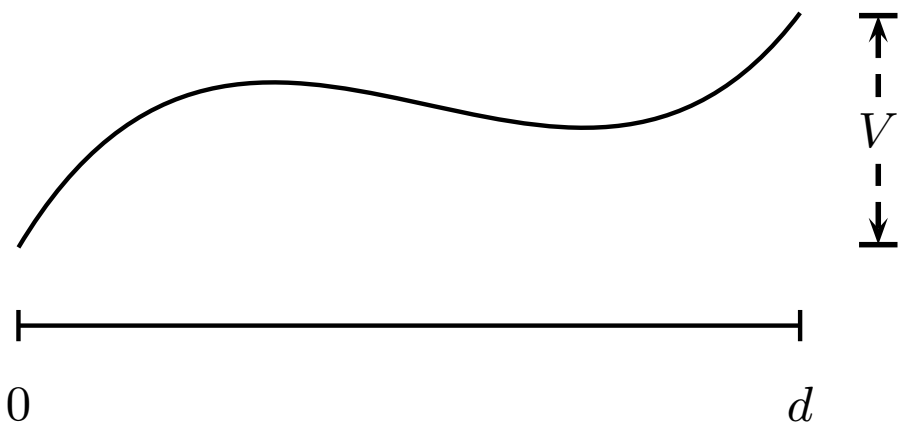
Trajectories

Definition

a trajectory t is a pair (d, g), where $d \in D$ is the duration and

$$g : [0, d] \to V \text{ or } f : [0, \infty) \to V$$

the image of $[0, d]$ ($[0, \infty)$) under g is its range $\mathtt{ran}(d, g)$



Durations

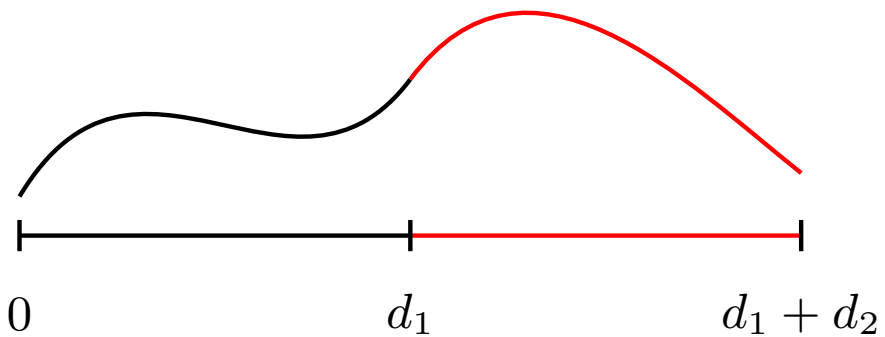
D has to fulfill some laws

- $(D, +, 0)$ cancellative commutative monoid
- order on D: $x \leq y \Leftrightarrow_{df} \exists z \,.\, x + z = y$
  - 0 least element
  - 0 indivisible, i.e., $x + y = 0 \Leftrightarrow x = y = 0$
- $\infty \in D$ possible
  - greatest element
  - not cancellative
- examples are $\mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{R} \times \mathbb{N} \ldots$

Composition of Trajectories

$$(d_1, g_1) \cdot (d_2, g_2) =_{df} \begin{cases} (d_1 + d_2, g) & \text{if } d_1 \neq \infty \wedge g_1(d_1) = g_2(0) \\ (d_1, g_1) & \text{if } d_1 = \infty \\ \text{undefined} & \text{otherwise} \end{cases}$$

with $g(x) = g_1(x)$ for all $x \in [0, d_1]$ and $g(x + d_1) = g_2(x)$ for all $x \in [0, d_2]$ or $x \in [0, \infty)$

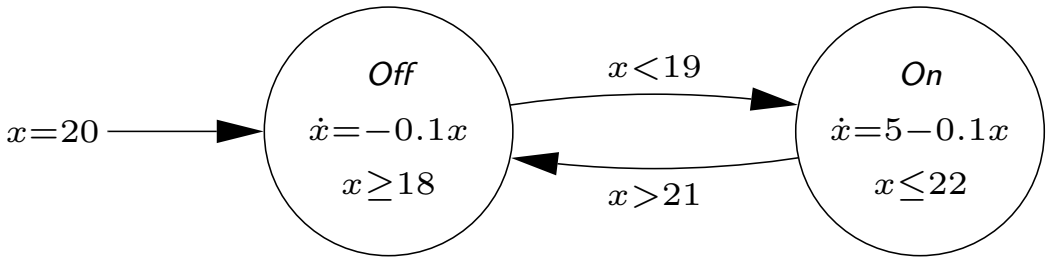


case of a zero-length trajectory

- $(0, g_1) \cdot (d_2, g_2) = (d_2, g_2)$ if $g_1(0) = g_2(0)$, otherwise undefined
- $(d_2, g_2) \cdot (0, g_1) = (d_2, g_2)$ if $d_2 \neq \infty$ and $g_2(d_2) = g_1(0)$

Connection to Hybrid Automata

a trajectory can model a run of a hybrid automaton



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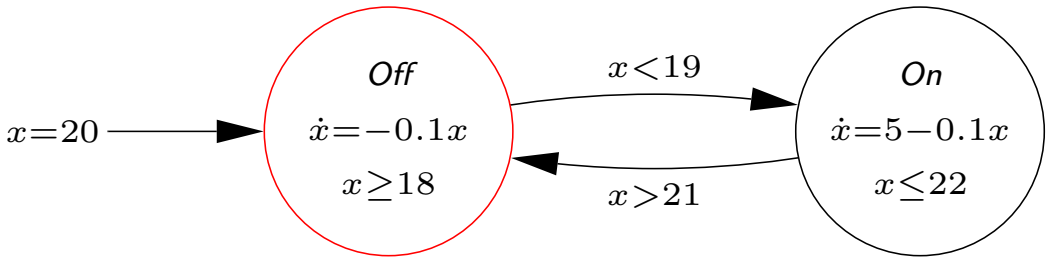

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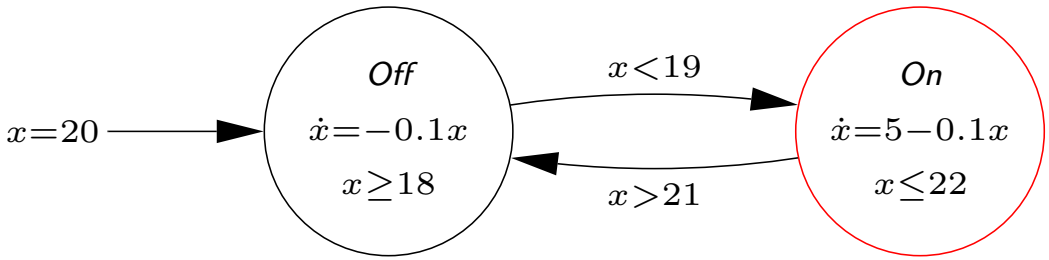

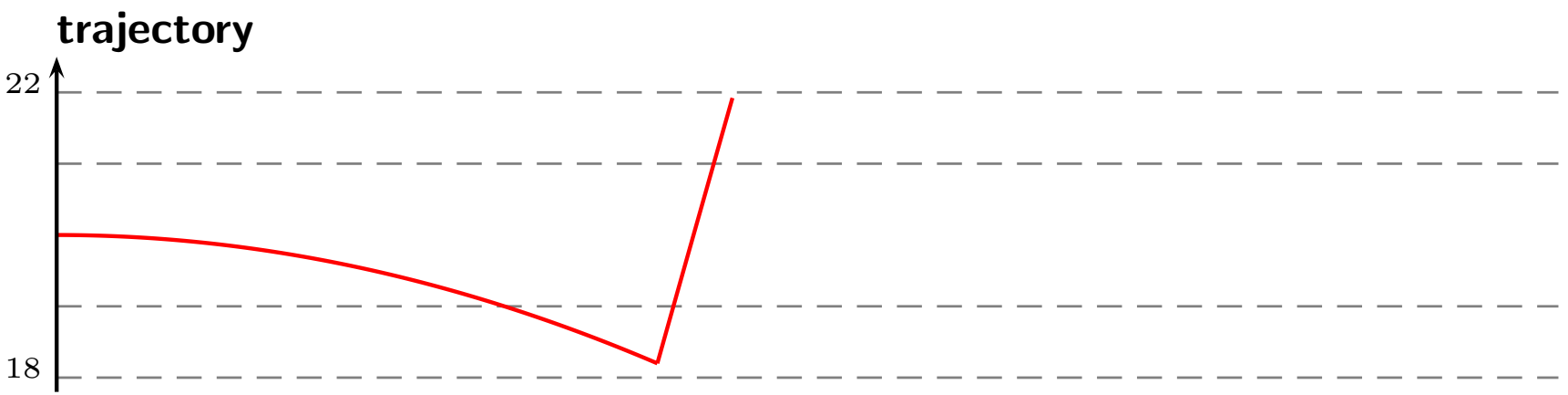

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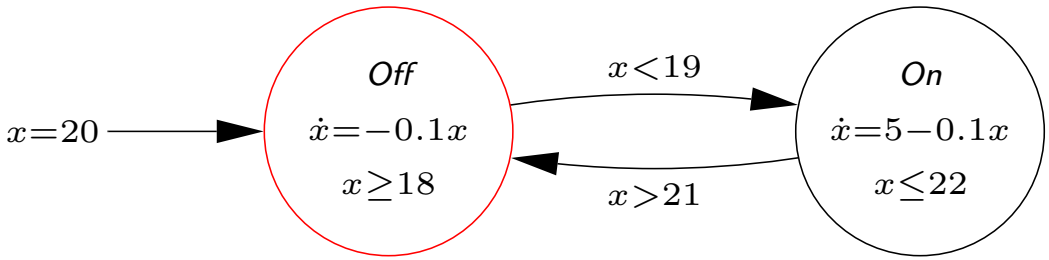

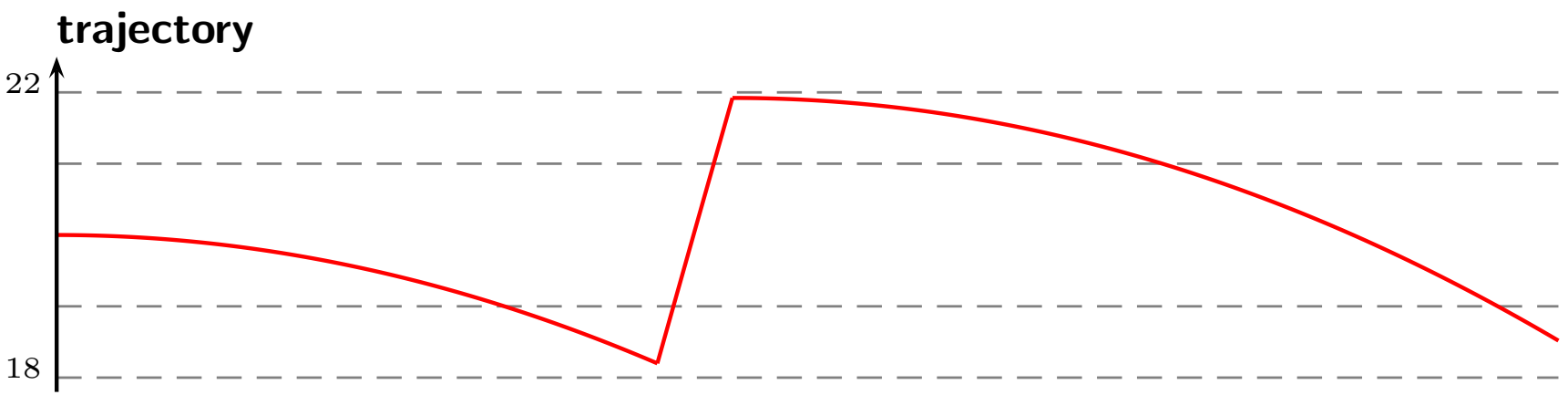

Getting Algebraic

the algebraic model of regular events is Kleene algebra

Definition

a Kleene algebra is a tuple $(K, +, 0, \cdot, 1, ^*)$ with

- $(K, +, 0)$ idempotent commutative monoid
- $(K, \cdot, 1)$ monoid
- multiplication is distributive
- 0 is an annihilator, $0 \cdot a = 0 = a \cdot 0$
- $*$ satisfies unfold and induction axioms

$+ \leftrightarrow$ choice
$\cdot \leftrightarrow$ sequential composition
$* \leftrightarrow$ finite iteration
$0 \leftrightarrow$ abort
$1 \leftrightarrow$ skip

Choice, Composition and neutral elements

- choice between trajectories is realised by set union over sets of trajectories (also called processes)
- the empty set is neutral element
- composition is lifted pointwise to processes

$$A \cdot B =_{df} \{a \cdot b \,|\, a \in A, b \in B\}$$

- the set of all trajectories with duration 0 (denoted by $\mathbb{1}$) is the neutral element

the algebra of hybrid systems $(\mathcal{P}(\mathrm{TRA}), \cup, \emptyset, \cdot, \mathbb{1}, ^*)$ is nearly a Kleene algebra (TRA is the set of all trajectories)

But

$$A \cdot \emptyset \neq \emptyset$$

Weak Kleene Algebra

Definition

a weak Kleene algebra is a Kleene algebra where 0 is only left annihilator $(0 \cdot a = 0)$

Remark

- relaxation allows to have infinite elements [Möller04]

$$\inf a = a \cdot 0 \qquad \text{fin}\, a = a - \inf a$$

- weak Kleene algebra behaves nearly like Kleene algebra
- adding infinite iteration yields weak omega algebra [Cohen00]
- adding tests to model assertions and guards [Kozen97]
- adding domain/codomain [DesharnaisMöllerStruth03/Möller04]
- in some situations one even needs no right-distributivity law ($\rightarrow$ left Kleene algebra)

Remarks on the Algebra of Hybrid Systems

- similarities to function spaces (linear algebra)
- if $D = \{0, 1\}$ then the algebra of hybrid systems is equivalent to relations
- it is even a weak quantale (Boolean complements can be defined)
- jumps at composition points possible
- restricted form of composition

$$A \frown B = (\mathsf{fin}\, A) \cdot B$$

the second trajectory is reached

Tests

(weak) Kleene algebra with tests

- $\mathsf{test}(S) \subseteq [0,1]$ Boolean subalgebra
- $0, 1 \in \mathsf{test}(S)$
- $p + q = p \sqcup q$, $p \cdot q = p \sqcap q$
- $\neg$ denotes complementation in $\mathsf{test}(S)$

- $\mathsf{test}(\mathrm{PRO}) = \{(0, g) \mid g(0) = v, v \in S \subseteq V\}$
  trajectories with duration 0
- $P \in \mathsf{test}(\mathrm{PRO}) \;\Rightarrow\; P \sqcap A \cdot B = (P \sqcap A) \cdot (P \sqcap B)$

Domain

(weak) Kleene algebra with Domain

- $\mathsf{dom} : S \to \mathsf{test}(S)$

  $$a \leq \mathsf{dom}(a) \cdot a \ , \qquad \mathsf{dom}(p \cdot a) \leq p \ , \qquad \mathsf{dom}(a \cdot \mathsf{dom} b) \leq \mathsf{dom}(a \cdot b)$$

  $(a, b \in S, p \in \mathsf{test}(S))$

- in PRO domain characterises starting points

  $$\mathsf{dom}\, A = \{(0, g(0)) \mid (d, g) \in A\}$$

Safety and Liveness

Safety: "something bad will never happen" [Lamport77]

- conservative in the sense of avoiding bad states
- e.g. do nothing
- something is true forever

Liveness: "something good will eventually happen" [Lamport77]

- progressiv in the sense of reaching good states
  or the system will never stop

Algebraic Safety and Liveness

Examples for Range-Restriction Operators

- P will be reached

$$\Diamond P =_{df} \mathsf{F} \cdot P \cdot \top$$

  set of all trajectories, where the range is within P at some point

- P is guaranteed

$$\Box P =_{df} \overline{\Diamond \neg P}$$

  set of all trajectories, where the range is complete in P
  needs complementation on underlying structure

where $\top =_{df} \mathrm{TRA}$ und $\mathsf{F} =_{df} \mathrm{fin}\,(\mathrm{TRA})$

Basic Properties

- $\Box p \sqcap a \cdot b = (\Box p \sqcap a) \cdot (\Box p \sqcap b)$
- $\Diamond p \sqcap a \cdot b = (\Diamond p \sqcap a) \cdot b + \mathsf{fin}\, a \cdot (\Diamond p \sqcap b)$
- $(\Box p) \cdot (\Box p) = \Box p$

for $p \in \mathsf{test}(K)$ and $a, b \in K$

More Modal Operators

there are several useful modal operators for the algebra of hybrid system for example

- the standard modal operators of Kleene algebra [DesharnaisMöllerStruth03]
- relaxation from tests to general elements

$$\langle a\rangle b =_{df} a \cdot b \cdot a, \qquad [a]b =_{df} \overline{\langle a\rangle \overline{b}}$$

- residuals and detachments of quantales ("there is a trajectory t in S ...")
- ...

all these operators are used in different situation to specify safety and liveness on an algebraic level

Overview of our work

- **What we have done**
- **What we do**
- **What we will do**

What we have done

- build an algebra of hybrid systems
- show basic properties
- describe and use the Duration Calculus [RavnHoareZhou91] in an algebraic setting
- characterise different useful modal operators in the setting, including the one of vonKarger, Sintzoff, ...

(Dis)Advantages of What we have done

- create a "uniform" basis
- algebraic structures like Kleene algebra are well known
- algebra allows easy calculations
- but sometimes domain-knowledge is needed
- not easy to understand (especially for non-computer scientists)
- aid of computer seems feasible

What we do

- adapt logics to hybrid systems
  there are algebraic versions of
  - Hoare Logic [Kozen97,MöllerStruth06]
  - LTL [DesharnaisMöllerStruth04]
  - CTL and CTL* [MöllerHöfnerStruth06]
  - Neighbourhood Logic of Zhou and Hansen [Höfner06]

  due to the algebraic versions which also use Kleene algebra, it should be possible
- there are notions of dynamic systems in relation algebra [ScolloFrancoManca06]
  can this be adopted/generalised to our framework?
- use of theorem provers (Prover9) [HöfnerStruth07]

(Dis)Advantages of What we do

- knowlegde transfer
  in CTL, CTL* there are notions of liveness, safety, ...
- use of standard terminology
- support of computers

What we will do

- discuss Zeno Effects
- bring game theory into play
  (first steps done in [Sintzoff04])