

f-Generated Kleene Algebra

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Motivation

- Kleene star sometimes not adequate
 - only iteration
 - no modification of iterated element

- idea: parametrise iteration with function f

- application: hybrid systems

- example: oscillation with damping of physical systems

Kleene Star

- (finite) iteration characterised by

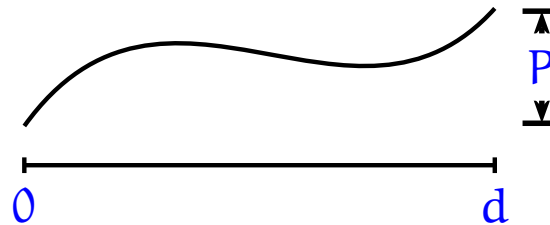
$$1 + a \cdot a^* \leq a^*$$

$$1 + a \cdot c \leq b \Rightarrow a^* \cdot c \leq b$$

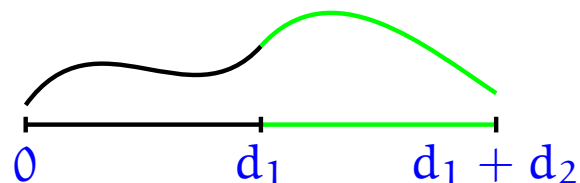
- a^* least fixed point of $\lambda x. 1 + a \cdot x$

Hybrid Systems

- *trajectories* are (pairs (d, g))
with *duration* $d \in \mathbb{R}$ and *behaviour* $g : [0, d] \rightarrow P$



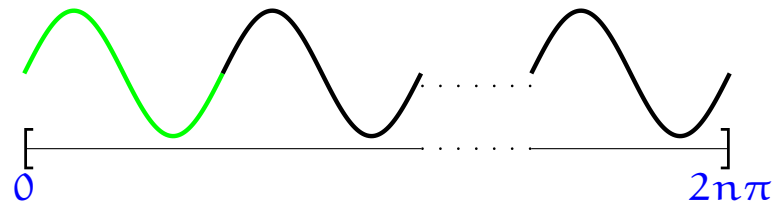
- composition of trajectories:



- sets of trajectories form a Kleene algebra [HöfnerMöller06]

Example: Oszillator

- free oscillator



- iteration over one period sinus function

- but: how to model damping?

e.g., $t_{\sin} =_{df} \{(2\pi, \frac{\sin x}{5x})\}$

f-Generated Kleene Algebra

- idea: extend $\lambda x. 1 + a \cdot x$ to $\lambda x. 1 + a \cdot f(x)$

Definition 1 *homomorphism* $f : S \rightarrow S$ on monoid (S, \circ, Id)

- $f(a \circ b) = f(a) \circ f(b)$ and $f(\text{Id}) = \text{Id}$

Definition 2 *f-generated Kleene algebra*

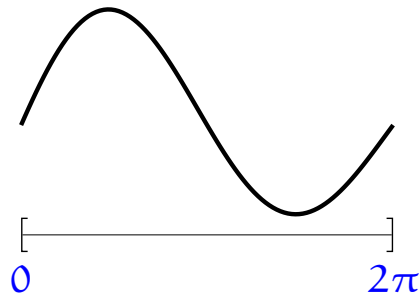
- $(S, +, \cdot, 0, 1)$ idempotent semiring
- $f : S \rightarrow S$ homomorphism w.r.t. $+$ and \cdot
- $1 + a \cdot f(a^{f*}) \leq a^{f*}$ and $1 + a \cdot f(b) \leq b \Rightarrow a^{f*} \leq b$

Properties

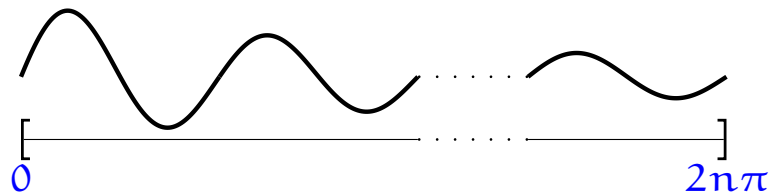
- $a^{f^*} = 1 + a \cdot f(1 + a \cdot f(1 + a \cdot f(1 + \dots)))$
 $1 + a + a \cdot f(a) + a \cdot f(a) \cdot f(f(a)) + \dots$
- a^{f^*} least prefixed point of $\lambda x. 1 + a \cdot f(x)$
- $1 + a \cdot f(a^{f^*}) \leq a^{f^*}$ can be strengthened to an equation
- no dual laws, since $a \cdot a^{f^*} \neq a^{f^*} \cdot a$

Example: Damped Oscillator

- let $t_{\sin} =_{df} \{(2\pi, \frac{\sin x}{5x})\}$



- e.g. $f((d, g)) =_{df} (d, \frac{g}{2})$ and t_{\sin}^{f*} yields



Remarks

- t_{sin}^* just repeats t_{sin} a finite number of times
- $t_{\text{sin}}^{f^*}$ repeats and modifies t_{sin}
- allows scaling of trajectories
- Zeno effects possible: $f((d, g(x))) = (\frac{d}{2}, g(\frac{x}{2}))$

Properties

- in functional programming

$$a^{f^*} = \text{sum scan } [1, a, f(a), f^2(a), \dots]$$

- some properties of star are lost, e.g.,
 - isotony holds
 - $a \cdot (b \cdot a)^{f^*} \neq (a \cdot b)^{f^*} \cdot a$
- f-generated Kleene algebra with tests possible
(analogue to [Kozen97])

Conclusion

- Kleene star sometimes not adequate
- parametrised iteration introduced
- f-generated Kleene star modifies the iterated element
- useful for describing hybrid systems

Outlook

- interaction of f and f^*
- connection to functional programming
- weaken the finite iteration of lazy Kleene algebra [Möller06]
- weaken the infinite iteration [Cohen00]
- weaken the strong iteration of refinement algebra [vonWright02]
- investigate applications in hybrid systems